Some Fast Solvers for Poroelastic Model with Applications in Brain Edema Simulation

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Outline

1. Biot Equations
2. Solvers based on Stabilized FE Discretizations
3. Algorithms based on a Multiphysics Reformulation
   - Numerical Experiments
4. Brain Edema Simulation
Biot Equations

Poroelastic models is for describing the phenomenon when elastic material is saturated in fluids. The governing equations for poroelasticity is the so-called Biot equations:

\[
\begin{align*}
-\text{div}(\mu[\nabla u + \nabla u^T]) - \nabla \lambda \text{div} u + \alpha \nabla p &= f \quad \forall x \in \Omega, \\
(c_0 p + \alpha \text{div} u)_t - \text{div} K(\nabla p - \rho_f g) &= Q_s. \quad \forall x \in \Omega,
\end{align*}
\]

subject to suitable BCs and ICs. \(u\): displacement of elastic body; \(p\): fluid pressure. The first equation is the momentum equation,

\[-\text{div}\sigma = f, \quad \text{with} \quad \sigma = \lambda \text{tr}(\varepsilon(u)) + 2\mu \varepsilon(u) - \alpha pl.\]

The second equation is the mass conservation equation. Inherently, fluid filtration velocity satisfies Darcy’s law:

\[q = -K \nabla p.\]
We consider partial Dirichlet and partial Neumann boundary condition.

\[ \partial \Omega = \Gamma_d \cup \Gamma_t \quad \text{and} \quad \partial \Omega = \Gamma_p \cup \Gamma_f. \]

\( \Gamma_d \): Dirichlet boundary for \( u \); \( \Gamma_t \): Neumann boundary for \( u \);
\( \Gamma_p \): Dirichlet boundary for \( u \); \( \Gamma_f \): Neumann boundary for \( p \).

The BCs are

\[
\begin{cases}
  u = 0 & \text{on } \Gamma_d, \\
  \sigma(u)n - \alpha pn = h & \text{on } \Gamma_t, \\
  p = 0 & \text{on } \Gamma_p, \\
  (\nabla p - \rho_f g) \cdot n = g_2 & \text{on } \Gamma_f.
\end{cases}
\]

(2)

ICs: \( u(0) = u_0 \), \quad \text{and} \quad p(0) = p_0. \)
Applications

Different components of a brain and a simplified FE poroelastic model of brain edema.
Applications Areas

Main application areas:
1. Geomechanics, including soil mechanics, rock mechanics. For example, for describing earthquakes (dynamic Biot equations).
2. Biomechanics, including tissue mechanics, and mechanical models of brain swelling and cancellous bones.
3. Material sciences, including deformation models of gels, polymer materials (saturated in various solvents).
Time Scheme and the Resulting Differential Operator

The backward Euler scheme:

$$-(\text{div} \mathbf{u}^{n+1} - \text{div} \mathbf{u}^n) - c_0(p^{n+1} - p^n) + \Delta t \text{div}(K \nabla p^{n+1}) = -\Delta t Q^{n+1}_s.$$ 

The resulting differential operator reads as:

$$\mathcal{M} = \begin{bmatrix} -\mu \Delta - (\mu + \lambda) \nabla \text{div} & \text{grad} \\ -\nabla \text{div} & -c_0 I + \theta \Delta \end{bmatrix}. \quad (3)$$

Here, $\theta = K \Delta t$, $\alpha$ is set to be 1 for simplicity. $\mu$ and $\lambda$ are the Lamé constants, in terms of Young’s module $E$ and Poisson ratio $\nu$:

$$\lambda = \frac{E \nu}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{E}{2(1 + \nu)}. \quad (4)$$
Introduction of Saddle Point Problems and Preconditioners

A general form of saddle point system reads as

\[ M = \begin{bmatrix} A & B^t \\ B & -D \end{bmatrix}. \]

Wellposedness: (1) Operator \( B \) needs to be a surjective, which is equivalent to the inf-sup condition. (2) Operator \( A \) needs to be invertible in \( \text{Ker}(B) \). (For example, if \( A \) is elliptic, then it is sufficient to guarantee the invertibility).

**Example 1.** Stationary Stokes operator with zero Dirichlet boundary condition:

\[ \mathcal{M} = \begin{bmatrix} -\nu \Delta & \nabla \\ -\nabla & 0 \end{bmatrix} : (H^1_0(\Omega))^n \times L^2_0(\Omega) \leftrightarrow (H^{-1}(\Omega))^n \times L^2_0(\Omega). \]
Saddle Point Problems and Preconditioners...

**Example 2.** The discretized Biot operator (3).

**Preconditioning:** For a general linear system $Ax = b$, we design an approximation of $A$, say $P$, then solve

$$P^{-1}Ax = P^{-1}b.$$ 

In preconditioners for saddle point problems, one needs to approximate $A$ and the Schur complement: $S = BA^{-1}B^t + D$.

Preconditioner for stationary Stokes operator:

$$P = \begin{bmatrix} -\nu \Delta & 0 \\ 0 & \frac{1}{\nu} I \end{bmatrix} : (H^1_0(\Omega))^n \times L^2_0(\Omega) \mapsto (H^{-1}(\Omega))^n \times L^2_0(\Omega).$$

Discretely, $-\nabla_h \cdot (-\Delta_h)^{-1} \nabla_h \approx I_h$, (by noting that $-\Delta = \text{div}(\nabla \cdot )$ if *inf-sup* stable discretization is used. The condition number $\kappa(P_h^{-1}M_h)$ is bounded by a number independent of $h$ and $\nu$. 
The differential operator form of $\mathcal{M}$ in 2D is

$$
\begin{bmatrix}
-(2\mu + \lambda)\partial_x^2 - \mu \partial_y^2 & -(\mu + \lambda)\partial_y \partial_x & \partial_x \\
-(\mu + \lambda)\partial_x \partial_y & -(2\mu + \lambda)\partial_y^2 - \mu \partial_x^2 & \partial_y \\
-\partial_x & -\partial_y & -(s - \theta(\partial_x^2 + \partial_y^2))
\end{bmatrix}.
$$

The Fourier mode $\hat{\mathcal{M}}$ and the Fourier mode $\hat{\mathcal{S}}$ are

$$
\hat{\mathcal{M}} = \begin{bmatrix}
(2\mu + \lambda)k^2 - \mu l^2 & (\mu + \lambda)kl & ik \\
(\mu + \lambda)kl & (2\mu + \lambda)l^2 - \mu k^2 & il \\
-ik & -il & -s + \theta(k^2 + l^2)
\end{bmatrix},
$$

$$
\begin{bmatrix}
-ik & -il
\end{bmatrix} \begin{bmatrix}
(2\mu + \lambda)k^2 + \mu l^2 & (\mu + \lambda)kl \\
(\mu + \lambda)kl & (2\mu + \lambda)l^2 + \mu k^2
\end{bmatrix}^{-1} \begin{bmatrix}
ik \\
il
\end{bmatrix}
+ s - \theta(k^2 + l^2) = \frac{1}{2\mu + \lambda} + [s - \theta(k^2 + l^2)].
$$

$$
\mathcal{S} = \frac{1}{2\mu + \lambda} l + D.
$$
Weak Forms

\[ \mathbf{V} := \{ \mathbf{v} \in (H^1(\Omega))^d : \mathbf{v}|_{\Gamma_d} = \mathbf{0} \}, \]
\[ M := \{ p \in H^1(\Omega), \ p|_{\Gamma_p} = 0 \}. \]

By integration by parts:

\[ \int_{\Omega} \nabla p \cdot \mathbf{v} = - \int_{\Omega} p \text{div} \mathbf{v} + \int_{\partial\Omega} p \mathbf{v} \cdot \mathbf{n}, \]

because

\[ \text{div}(p \mathbf{v}) = \nabla p \cdot \mathbf{v} + p \text{div} \mathbf{v}. \]

The boundary term vanishes, then \( \int_{\Omega} \nabla p \cdot \mathbf{v} = - \int_{\Omega} p \text{div} \mathbf{v} \).

Similarly, integration parts are applied for the linear elasticity operator (and the mass conservation equation):

\[ - \int_{\Omega} \text{div} \sigma \cdot \mathbf{v} = \int_{\Omega} \sigma : \nabla \mathbf{v} + \int_{\partial\Omega} (\sigma \mathbf{n}) \cdot \mathbf{v}. \]
Weak Forms for Biot Equations

The weak form reads as: find $u \in V, p \in M$, such that

$$
\begin{align*}
\left\{ 
\begin{array}{ll}
    a(u, v) + b(v, p) = (f, v) & \forall v \in V, \\
    b(u, q) - d(p, q) = (Q_s, q) & \forall q \in M.
\end{array}
\right.
\end{align*}
$$

(5)

$$
a(u, v) = \int_{\Omega} 2\mu \varepsilon(u) : \varepsilon(v) + \lambda \text{div} u \text{div} v d\Omega,
$$

(6)

$$
\varepsilon(u) : \varepsilon(v) = \sum_{i=1}^{d} \sum_{j=1}^{d} \varepsilon_{ij}(u) \varepsilon_{ij}(v),
$$

$$
d(p, q) = \int_{\Omega} c_0 pq + \theta \nabla p \cdot \nabla q d\Omega,
$$

(7)

$$
b(v, q) = -\int_{\Omega} \text{div} v \ q \ d\Omega.
$$

(8)
Stabilized Finite Element Approximation

Due to the Dirichlet BC of pressure and to guarantee the monotonicity, the stabilized bilinear form for reaction-diffusion operator becomes C. Rodrigo, F. Gaspar, X. Hu, L. Zikatanov

\[
\tilde{d}_h(p_h^{n+1}, q_h) = d(p_h, q_h) + \epsilon \frac{h^2}{\Delta t} \frac{1}{2\mu + \lambda} \int_{\Omega} \nabla p_h^{n+1} \nabla q_h. \tag{9}
\]

Here, the second term is the stabilization term. Correspondingly, the RHS of the second equation is

\[
(\tilde{Q}_s, q_h) = (Q_s, q_h) + \epsilon \frac{h^2}{\Delta t} \frac{1}{2\mu + \lambda} \int_{\Omega} \nabla p_h^n \nabla q_h.
\]

It is suggested that \(\epsilon = \frac{1}{6}\) for inf-sup stable FEs (e.g. Mini and Taylor-Hood elements) and \(\epsilon = \frac{1}{4}\) for equal-order FEs.
Denote $A$, $B$, $D$ as the sub-block matrices in the saddle point system.

$$P_1 = \begin{bmatrix} P_A & 0 \\ 0 & -P_S \end{bmatrix}, \quad P_2 = \begin{bmatrix} P_A & 0 \\ B & -P_S \end{bmatrix}, \quad P_3 = \begin{bmatrix} P_A & B^t \\ 0 & -P_S \end{bmatrix}.$$  

We set $P_A = A$. The approximation of $S = BA^{-1}B^t + D$ reads as

$$P_S = \frac{1}{(2\mu + \lambda)} M_p + D. \quad (10)$$  

Here, $M_p$, which corresponds to the identity operator in the pressure space, is the pressure mass matrix.
Analysis

Let $A_0(i, j) = \langle \nabla b_i, \nabla b_j \rangle$ be the matrix corresponding to the vector Laplacian.

$$\alpha_A(A_0u_h, u_h) \leq (Au_h, u_h) \leq (2\mu + \lambda)(A_0u_h, u_h).$$

$$\frac{1}{2\mu + \lambda}(A_0^{-1}u, u) \leq (A^{-1}u, u) \leq \frac{1}{\alpha_A}(A_0^{-1}u, u), \quad \forall u.$$

$$\frac{\frac{1}{2\mu + \lambda}(BA_0^{-1}B^tp,p)+(Dp,p)}{\left(\frac{1}{2\mu + \lambda}M_p+D\right)p,p} \leq \frac{((BA^{-1}B^t+D)p,p)}{\left(\frac{1}{2\mu + \lambda}M_p+D\right)p,p}$$

$$\leq \frac{1}{\alpha_A}(BA_0^{-1}B^tp,p)+(Dp,p)}{\left(\frac{1}{2\mu + \lambda}M_p+D\right)p,p}. \quad (11)$$
Main Results

As we use inf-sup stable Mini elements, \( \exists \beta > 0 \), independent of \( h \), such that (cf. H. Elman’s book, 2005 or 2014)

\[
\beta^2 \leq \frac{(BA^{-1}_0 B^t \mathbf{p}, \mathbf{p})}{(M \rho \mathbf{p}, \mathbf{p})} \leq 1.
\]

Plugging the above inequality into (11), we see that

**Theorem**

*For the Biot problem discretized by the stabilized Mini elements, if exact elliptic solvers are applied in \( P_2^{-1} \) (or \( P_3^{-1} \)), the eigenvalues of the preconditioned system are either 1 or have uniform lower and upper bounds independent of mesh refinement and physical parameters.*
### Numerical Experiments

<table>
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<tr>
<th>DOFs</th>
<th>Nx</th>
<th>No Pre</th>
<th>PGMRES</th>
<th>U</th>
<th>AAU</th>
<th>VU</th>
<th>IPGMRES</th>
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**Table:** No. of iters, with exact (U, AAU, and VU) Poisson solvers and inexact Poisson solvers (IPGMRES). $E = 1000$, $\nu = 0.3$, and all the other parameters are equal to 1.

<table>
<thead>
<tr>
<th>DOFs</th>
<th>Nx</th>
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<th>PGMRES</th>
<th>U</th>
<th>AAU</th>
<th>VU</th>
<th>IPGMRES</th>
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<td>5</td>
<td>8</td>
<td>515</td>
</tr>
</tbody>
</table>

**Table:** No. of iters, with exact (U, AAU, and VU) Poisson solvers and inexact Poisson solvers (IPGMRES). $E = 1000$, $\nu = 0.49$ and all the other parameters are equal to 1.
Variables are kept as less as possible.

Preconditioners are robust with respect to the mesh refinement, physics parameters (in particular, the Poisson ratio).

Preconditioned Uzawa, Variable-relaxation Uzawa, and Anderson accelerated Uzawa are not as efficient as PGMRES method.

Bad aspect: no good solver for the linear elasticity operator. For inverting the linear elasticity operator, one can try the two-level overlapping Schwarz method.

Revisit the Biot Equations

In the Biot equations,

$$-\text{div}(\mu[\nabla u + \nabla u^T]) - \nabla \lambda \text{div} u$$

is the classical linear elasticity operator. This operator can be reformulated as a mixed operator. Noting that if $\lambda$ and $\mu$ are constants, we have the identity:

$$-\text{div}(\mu[\nabla u + \nabla u^T]) - \nabla \lambda \text{div} u = -\mu \Delta u - (\mu + \lambda) \nabla \text{div} u. \quad (12)$$

One can introduce more intermediate variables (e.g. J. Lee, K. Mardal, R. Winther SISC2017) to get a $3 \times 3$ or $4 \times 4$ saddle point system.
A Multiphysics Reformulation of the PDE Model

We introduce a new variable:

\[ \xi = \alpha \rho - \lambda \text{div } u, \]

One can call \( \xi \) as "total pressure".

Then, the new form of Biot problem is:

\[
-2\mu \text{div}(\varepsilon(u)) + \nabla \xi = f, \tag{13}
\]

\[
-\text{div } u - \frac{1}{\lambda} \xi + \frac{\alpha}{\lambda} \rho = 0, \tag{14}
\]

\[
((c_0 + \frac{\alpha^2}{\lambda}) \rho - \frac{\alpha}{\lambda} \xi)_t - K \text{div } (\nabla \rho - \rho_f g) = Q_s. \tag{15}
\]
Based on the new formulation, the weak form reads as: find $(\mathbf{u}, \xi, p) \in V \times W \times M$ such that $\forall \mathbf{v} \in V, \forall \varphi \in W, \forall \psi \in M,$

$$2\mu(\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v})) - (\xi, \text{div} \mathbf{v}) = (f, \mathbf{v}) + <h, \mathbf{v}>_{\Gamma_t},$$

$$-(\text{div} \mathbf{u}, \phi) - \frac{1}{\lambda}(\xi, \phi) + \frac{\alpha}{\lambda}(p, \phi) = 0,$$

$$\left(\left((c_0 + \frac{\alpha^2}{\lambda})p - \frac{\alpha}{\lambda}\xi\right)_t, \psi\right) + K(\nabla p, \nabla \psi) = (Q_s, \psi) + <g_2, \varphi>_{\Gamma_f}.$$

One can move $\frac{\alpha}{\lambda}(p, \phi)$ to the RHS, then the first two eqns are the mixed form of linear elasticity.
We use conforming Finite element discretization: \( \mathbf{V}_h \subset \mathbf{V}, \) \( M_h \subset M \) and \( W_h \subset W. \)

\[
\mathbf{V}_h := \{ \mathbf{v}_h \in C(\bar{\Omega}); \mathbf{v}_h = \mathbf{0}|_{\Gamma_d}; \mathbf{v}_h|_K \in P_2(K), \ \forall K \in T_h \},
\]

\[
W_h := \{ \xi_h \in L^2(\Omega); \xi_h|_K \in P_1(K), \ \forall K \in T_h \},
\]

\[
M_h := \{ \rho_h \in C(\bar{\Omega}); \rho_h = 0|_{\Gamma_p}; \rho_h|_K \in P_1(K), \ \forall K \in T_h \}.
\]
Algorithm 1 A Coupled Algorithm

Evaluate $u^0_h, p^0_h$, and $\xi^0_h$ by $\xi^0_h = \alpha p^0_h - \lambda \text{div } u^0_h$;
For $n = 0, 1, 2, \ldots$
Solve for $(u^{n+1}_h, \xi^{n+1}_h, p^{n+1}_h) \in V_h \times M_h \times W_h$ such that:

$$2\mu(\varepsilon(u^{n+1}_h), \varepsilon(v_h)) - (\xi^{n+1}_h, \text{div } v_h) = (f^n, v_h) + \langle h^n, v_h \rangle_{\Gamma_t},$$

$$-(\text{div } u^{n+1}_h, \varphi_h) - \frac{1}{\lambda}(\xi^{n+1}_h, \varphi_h) + \frac{1}{\lambda}(\alpha p^{n+1}_h, \varphi_h) = 0,$$

$$\left(\left(((c_0 + \frac{\alpha^2}{\lambda}) p^{n+1}_h - \frac{\alpha}{\lambda} \xi^{n+1}_h)/\Delta t, \psi_h\right) + K(\nabla p^{n+1}_h, \nabla \psi_h) = (Q_s, \psi_h)ight) + \left(\left(((c_0 + \frac{\alpha^2}{\lambda}) p^n_h - \frac{\alpha}{\lambda} \xi^n_h)/\Delta t, \psi_h\right) + \langle g_2, \psi_h \rangle_{\Gamma_f}.\right.$$
A Decoupled Algorithm

**Algorithm 2 A Decoupled Algorithm**

Evaluate $\mathbf{u}_h^0$, $\mathbf{p}_h^0$, and $\xi_h^0$ by $\xi_h^0 = \alpha \mathbf{p}_h^0 - \lambda \text{div} \mathbf{u}_h^0$

For $n = 0, 1, 2, \ldots$

(i) Finding $(\mathbf{u}_h^{n+1}, \xi_h^{n+1}) \in \mathbf{V}_h \times M_h$ such that:

$$2\mu(\varepsilon(\mathbf{u}_h^{n+1}), \varepsilon(\mathbf{v}_h)) - (\xi_h^{n+1}, \text{div} \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) + < \mathbf{h}, \mathbf{v}_h >_{\Gamma_t},$$

$$- (\text{div} \mathbf{u}_h^{n+1}, \varphi_h) - \frac{1}{\lambda} (\xi_h^{n+1}, \varphi_h) = - \frac{1}{\lambda} (\alpha \mathbf{p}_h^n, \varphi_h).$$

(ii) \[
\left( \left( c_0 + \frac{\alpha^2}{\lambda} \right) \frac{\mathbf{p}_h^{n+1}}{\Delta t}, \psi_h \right) + K(\nabla \mathbf{p}_h^{n+1}, \nabla \psi_h) = (Q_s, \psi_h) \\
+ \left( \left( c_0 + \frac{\alpha^2}{\lambda} \right) \frac{\mathbf{p}_h^n}{\Delta t}, \psi_h \right) + \frac{\alpha}{\lambda} \left( \frac{\xi_h^{n+1} - \xi_h^n}{\Delta t}, \psi_h \right) + < g_2, \psi_h >_{\Gamma_f}.
\]
Let $\Omega = [0, 1] \times [0, 1]$, $\Gamma_1 = \{(1, y); 0 \leq y \leq 1\}$, $\Gamma_2 = \{(x, 0); 0 \leq x \leq 1\}$, $\Gamma_3 = \{(0, y); 0 \leq y \leq 1\}$, $\Gamma_4 = \{(x, 1); 0 \leq x \leq 1\}$, $T = 0.001$. We vary the Poisson ratio, fixed $E = 1000$, other perimeters are equal to 1. The force and the source terms are selected so that

$$Q_s = (-c_0 + 2\kappa) \sin(x + y)e^{-t} - \alpha(\cos x + \cos y)e^{-t},$$

$$f = (\lambda + 2\mu)e^{-t} \begin{pmatrix} \sin x \\ \sin y \end{pmatrix} + \alpha \cos(x + y)e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$p = \sin(x + y)e^{-t} \text{ on } \Gamma_j, \ j = 1, 3,$$

$$u_1 = \sin xe^{-t}, \quad u_2 = \sin ye^{-t} \text{ on } \Gamma_j, \ j = 1, 3,$$

$$\sigma n - \alpha pn = h \quad \text{ on } \Gamma_j, \ j = 2, 4,$$

$$\nabla p \cdot n = \cos(x + y)e^{-t}(n_1 + n_2) \quad \text{ on } \Gamma_j, \ j = 2, 4,$$

$$u = 0, \quad p = \sin(x + y) \quad \text{ in } \Omega \times \{t = 0\},$$

(16)
### Tests for $\nu$ (Coupled Algorithm)

**Table:** Rate of convergence of the coupled algorithm for $\nu = 0.3$

<table>
<thead>
<tr>
<th>Meshes</th>
<th>$H^1$ errors of $u$</th>
<th>Orders</th>
<th>$L^2$ &amp; $H^1$ errors of $\xi$</th>
<th>Orders</th>
<th>$L^2$ &amp; $H^1$ errors of $p$</th>
<th>Orders</th>
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<td>2.03</td>
<td>2.192e-2 &amp; 5.266</td>
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**Table:** Rate of convergence of the coupled algorithm for $\nu = 0.499$

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<th>Meshes</th>
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<th>Orders</th>
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<th>Orders</th>
<th>$L^2$ &amp; $H^1$ errors of $p$</th>
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<td>9.100e-4 &amp; 2.753e-2</td>
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<td>6.332 &amp; 1523</td>
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<td>2.287e-4 &amp; 1.386e-2</td>
<td>1.99 &amp; 0.99</td>
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<td>0.3809 &amp; 379.3</td>
<td>2.02 &amp; 1.00</td>
<td>1.432e-5 &amp; 3.482e-3</td>
<td>2.00 &amp; 1.00</td>
</tr>
</tbody>
</table>
### Tests for $\nu$ (Decoupled Algorithm)

#### Table: Rate of convergence of the decoupled algorithm for $\nu = 0.3$

<table>
<thead>
<tr>
<th>Meshes</th>
<th>$H^1$ errors of $u$</th>
<th>Orders</th>
<th>$L^2$ &amp; $H^1$ errors of $\xi$</th>
<th>Orders</th>
<th>$L^2$ &amp; $H^1$ errors of $p$</th>
<th>Orders</th>
</tr>
</thead>
<tbody>
<tr>
<td>2384</td>
<td>4.241e-5</td>
<td>2.03</td>
<td>2.192e-2 &amp; 5.266</td>
<td>2.06</td>
<td>2.285e-4 &amp; 1.386e-2</td>
<td>1.99</td>
</tr>
<tr>
<td>9536</td>
<td>1.049e-5</td>
<td>2.02</td>
<td>5.350e-3 &amp; 2.629</td>
<td>2.03</td>
<td>5.724e-5 &amp; 6.954e-3</td>
<td>2.00</td>
</tr>
<tr>
<td>38144</td>
<td>2.604e-6</td>
<td>2.01</td>
<td>1.318e-3 &amp; 1.313</td>
<td>2.02</td>
<td>1.432e-5 &amp; 3.482e-3</td>
<td>2.00</td>
</tr>
</tbody>
</table>

#### Table: Rate of convergence of the decoupled algorithm for $\nu = 0.499$

<table>
<thead>
<tr>
<th>Meshes</th>
<th>$H^1$ errors of $u$</th>
<th>Orders</th>
<th>$L^2$ &amp; $H^1$ errors of $\xi$</th>
<th>Orders</th>
<th>$L^2$ &amp; $H^1$ errors of $p$</th>
<th>Orders</th>
</tr>
</thead>
<tbody>
<tr>
<td>596</td>
<td>2.000e-2</td>
<td></td>
<td>26.38 &amp; 2963</td>
<td></td>
<td>9.100e-4 &amp; 2.753e-2</td>
<td></td>
</tr>
<tr>
<td>2384</td>
<td>3.720e-3</td>
<td>2.43</td>
<td>6.332 &amp; 1523</td>
<td>2.06</td>
<td>2.287e-4 &amp; 1.386e-2</td>
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</tr>
<tr>
<td>9536</td>
<td>6.875e-4</td>
<td>2.44</td>
<td>1.546 &amp; 759.8</td>
<td>2.03</td>
<td>5.728e-5 &amp; 6.954e-3</td>
<td>2.00</td>
</tr>
<tr>
<td>38144</td>
<td>1.236e-4</td>
<td>2.48</td>
<td>0.3809 &amp; 379.3</td>
<td>2.02</td>
<td>1.433e-5 &amp; 3.482e-3</td>
<td>2.00</td>
</tr>
</tbody>
</table>
Numerical Experiments

**Tests for $\kappa$ (Coupled Algorithm)**

**Table:** Rate of convergence of the coupled algorithm of for $\kappa = 10^{-2}$

<table>
<thead>
<tr>
<th>Meshes</th>
<th>$H^1$ errors of $u$</th>
<th>Orders</th>
<th>$L^2$ &amp; $H^1$ errors of $\xi$</th>
<th>Orders</th>
<th>$L^2$ &amp; $H^1$ errors of $p$</th>
<th>Orders</th>
</tr>
</thead>
<tbody>
<tr>
<td>2384</td>
<td>4.241e-5</td>
<td>2.03</td>
<td>2.192e-2 &amp; 5.266</td>
<td>2.357e-4 &amp; 1.406e-2</td>
<td>2.00 &amp; 1.01</td>
<td></td>
</tr>
<tr>
<td>9536</td>
<td>1.049e-5</td>
<td>2.02</td>
<td>5.351e-3 &amp; 2.629</td>
<td>5.908e-5 &amp; 6.986e-3</td>
<td>2.00 &amp; 1.01</td>
<td></td>
</tr>
<tr>
<td>38144</td>
<td>2.604e-6</td>
<td>2.01</td>
<td>1.318e-3 &amp; 1.313</td>
<td>1.478e-5 &amp; 3.486e-3</td>
<td>2.00 &amp; 1.00</td>
<td></td>
</tr>
</tbody>
</table>

**Table:** Rate of convergence of the coupled algorithm for $\kappa = 10^{-6}$

<table>
<thead>
<tr>
<th>Meshes</th>
<th>$H^1$ errors of $u$</th>
<th>Orders</th>
<th>$L^2$ &amp; $H^1$ errors of $\xi$</th>
<th>Orders</th>
<th>$L^2$ &amp; $H^1$ errors of $p$</th>
<th>Orders</th>
</tr>
</thead>
<tbody>
<tr>
<td>2384</td>
<td>4.241e-5</td>
<td>2.03</td>
<td>2.192e-2 &amp; 5.266</td>
<td>2.360e-4 &amp; 1.409e-2</td>
<td>2.00 &amp; 1.01</td>
<td></td>
</tr>
<tr>
<td>9536</td>
<td>1.049e-5</td>
<td>2.02</td>
<td>5.351e-3 &amp; 2.629</td>
<td>5.920e-5 &amp; 7.003e-3</td>
<td>2.00 &amp; 1.01</td>
<td></td>
</tr>
<tr>
<td>38144</td>
<td>2.604e-6</td>
<td>2.01</td>
<td>1.318e-3 &amp; 1.313</td>
<td>1.482e-5 &amp; 3.492e-3</td>
<td>2.00 &amp; 1.00</td>
<td></td>
</tr>
</tbody>
</table>
Numerical Experiments

Tests for $\kappa$ (Decoupled Algorithm)

**Table:** Rate of convergence of the decoupled algorithm for $\kappa = 10^{-2}$

<table>
<thead>
<tr>
<th>Meshes</th>
<th>$H^1$ errors of $u$</th>
<th>Orders</th>
<th>$L^2$ &amp; $H^1$ errors of $\xi$</th>
<th>Orders</th>
<th>$L^2$ &amp; $H^1$ errors of $p$</th>
<th>Orders</th>
</tr>
</thead>
<tbody>
<tr>
<td>2384</td>
<td>4.241e-5</td>
<td>2.03</td>
<td>2.192e-2 &amp; 5.266</td>
<td>2.06 &amp; 0.96</td>
<td>2.357e-4 &amp; 1.406e-2</td>
<td>1.99 &amp; 1.04</td>
</tr>
<tr>
<td>9536</td>
<td>1.049e-5</td>
<td>2.02</td>
<td>5.351e-3 &amp; 2.629</td>
<td>2.03 &amp; 1.00</td>
<td>5.909e-5 &amp; 6.986e-3</td>
<td>2.00 &amp; 1.01</td>
</tr>
<tr>
<td>38144</td>
<td>2.604e-6</td>
<td>2.01</td>
<td>1.318e-3 &amp; 1.313</td>
<td>2.02 &amp; 1.00</td>
<td>1.479e-5 &amp; 3.486e-3</td>
<td>2.00 &amp; 1.00</td>
</tr>
</tbody>
</table>

**Table:** Rate of convergence of the decoupled algorithm for $\kappa = 10^{-6}$

<table>
<thead>
<tr>
<th>Meshes</th>
<th>$H^1$ errors of $u$</th>
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<th>Orders</th>
<th>$L^2$ &amp; $H^1$ errors of $p$</th>
<th>Orders</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2.03</td>
<td>2.192e-2 &amp; 5.266</td>
<td>2.06 &amp; 0.96</td>
<td>2.360e-4 &amp; 1.409e-2</td>
<td>1.99 &amp; 1.04</td>
</tr>
<tr>
<td>9536</td>
<td>1.049e-5</td>
<td>2.02</td>
<td>5.351e-3 &amp; 2.629</td>
<td>2.03 &amp; 1.00</td>
<td>5.921e-5 &amp; 7.003e-3</td>
<td>2.00 &amp; 1.01</td>
</tr>
<tr>
<td>38144</td>
<td>2.604e-6</td>
<td>2.01</td>
<td>1.318e-3 &amp; 1.313</td>
<td>2.02 &amp; 1.00</td>
<td>1.483e-5 &amp; 3.492e-3</td>
<td>2.00 &amp; 1.00</td>
</tr>
</tbody>
</table>
Path of CSF Flow

Figure: Path of CSF (Cerebrospinal fluid) flow.
Brain Slice and FE Mesh

Figure: A brain slice and the FE mesh.
Boundary Conditions

- $\Gamma_1$ is the skull, the displacement along $\Gamma_1$ is zero:

$$u = 0, \text{ on } \Gamma_1.$$  

- When CSF flows out of the brain tissue, it is absorbed the SAS part. It needs to meet the balance of flow rate, i.e.,

$$K \nabla p \cdot n = c_b (\rho_{SAS} - p), \quad \text{on } \Gamma_1. \quad (17)$$

- $\Gamma_2$ is the ventricle wall, the total normal force from the tissue part needs to balance the fluid pressure:

$$(\sigma_s - \alpha p) \cdot n = -p \cdot n \quad \text{on } \Gamma_2. \quad (18)$$

The pressure at the ventricle wall is around

$$p = 1100 \text{ Pa} \quad \text{on } \Gamma_2. \quad (19)$$
## Physics Parameters

**Table:** Lists of the main mathematical symbols and the corresponding physics meanings.

<table>
<thead>
<tr>
<th>Syms</th>
<th>Physics meaning</th>
<th>Syms</th>
<th>Physics meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>pore pressure</td>
<td>( \mu )</td>
<td>shear modulus</td>
</tr>
<tr>
<td>( \nu )</td>
<td>Poisson ratio</td>
<td>( \alpha )</td>
<td>Biot coefficient</td>
</tr>
<tr>
<td>( c_0 )</td>
<td>specific storage term</td>
<td>( \kappa )</td>
<td>permeability of the brain</td>
</tr>
<tr>
<td>( \mu_f )</td>
<td>fluid viscosity</td>
<td>( Q_s )</td>
<td>sources and sinks</td>
</tr>
<tr>
<td>( u )</td>
<td>displacement</td>
<td>( n )</td>
<td>normal vector</td>
</tr>
<tr>
<td>( c_b )</td>
<td>outflow conductance</td>
<td>( \rho_b )</td>
<td>venous blood pressure</td>
</tr>
</tbody>
</table>
Parameter Values of the Baseline Model

Table: Parameter values (key parameters (red color) for the baseline model)

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_0$</td>
<td>$4.5 \times 10^{-7}$ Pa$^{-1}$</td>
<td>$\kappa_0$</td>
<td>$1.4 \times 10^{-9}$ mm$^2$</td>
</tr>
<tr>
<td>$c_b$</td>
<td>$3 \times 10^{-5}$ mm/min/Pa</td>
<td>$\alpha$</td>
<td>1</td>
</tr>
<tr>
<td>$\rho_{SAS}$</td>
<td>1070 Pa</td>
<td>$\nu_0$</td>
<td>0.35</td>
</tr>
<tr>
<td>$\mu_f$</td>
<td>$1.48 \times 10^{-5}$ Pa$\cdot$min</td>
<td>$E_0$</td>
<td>9010 Pa</td>
</tr>
</tbody>
</table>

Other key parameters: the absorption rate of the injured brain tissue is $Q_s = 9 \times 10^{-3}$ mm$^3$/min. At the normal state $Q_s = 0$; When infusing or in a swelling state, $Q_s \neq 0$. 
Pressure Distribution of the Normal State

Figure: Pressure distribution of the normal state of a brain (Left: our result; Right: by Li et al.)
FE Mesh for an Injured Brain

**Figure:** FE mesh for an injured brain.
Pressure and Displacement Distribution of an Injured Brain

Figure: Pressure and displacement distribution after TBI (Parameters are from the baseline model)
The Max Values of ICP and Displacement w.r.t. Time

**Figure:** The maximum values of ICP and tissue displacement as time evolves (after TBI, parameters are from the baseline model)
Data Sets for Testing the Effects of Parameters

**Table:** Data sets used to study the effects of physics parameters

<table>
<thead>
<tr>
<th></th>
<th>$E_0 = 9010$ Pa, $\nu_0 = 0.35$, $\kappa_0 = 1.4 \times 10^{-9}$ mm$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$ (Pa)</td>
<td>0.2$E_0$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.3</td>
</tr>
<tr>
<td>$\kappa$ (mm$^2$)</td>
<td>$1.4 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

We vary these values because: on one hand, it is very difficult to get the real values experimentally; on the other hand, there are big variations of parameters used in the literature.
The max ICP and Displacement Curves under Different $E$

**Figure:** The maximum values of pressure and displacement as time evolves. $E = 0.2E_0$ (left), $E = 10E_0$ (right)

**Table:** $u_{max}$ and $p_{max}$. Fixing $\nu = \nu_0$ and $\kappa = \kappa_0$.

<table>
<thead>
<tr>
<th>$E$</th>
<th>$\mu$</th>
<th>$1/\lambda$</th>
<th>$u_{max}$</th>
<th>$p_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E = 0.2E_0$</td>
<td>667</td>
<td>$6.42 \times 10^{-4}$</td>
<td>3.3 mm</td>
<td>3023 Pa</td>
</tr>
<tr>
<td>$E = 10E_0$</td>
<td>33370</td>
<td>$1.28 \times 10^{-5}$</td>
<td>0.0664 mm</td>
<td>3025 Pa</td>
</tr>
</tbody>
</table>
The max ICP and Displacement Curves under Different $\nu$

![Graphs showing maximum pressure and displacement over time for different values of $\nu$.](image)

**Figure:** The maximum values of pressure and displacement as time evolves. $\nu = 0.3$ (left), $\nu = 0.499$ (right)

**Table:** $u_{max}$ and $p_{max}$. Fixing $E = E_0$ and $\kappa = \kappa_0$

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$\mu$</th>
<th>$1/\lambda$</th>
<th>$u_{max}$</th>
<th>$p_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>3465</td>
<td>$1.9 \times 10^{-4}$</td>
<td>0.7356 mm</td>
<td>3025 Pa</td>
</tr>
<tr>
<td>0.499</td>
<td>3005</td>
<td>$6.67 \times 10^{-7}$</td>
<td>0.01218 mm</td>
<td>3025 Pa</td>
</tr>
</tbody>
</table>
The max ICP and Displacement Curves under Different $\kappa$

**Figure:** The maximum values of pressure and displacement as time evolves. $\kappa = 0.1\kappa_0$ (left), $\kappa = 10\kappa_0$ (right)

**Table:** $u_{\text{max}}$ and $p_{\text{max}}$. Fixing $E = E_0$ and $\nu = \nu_0$.

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$\mu$</th>
<th>$\frac{1}{\lambda}$</th>
<th>$u_{\text{max}}$</th>
<th>$p_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.1\kappa_0$</td>
<td>3337</td>
<td>$1.28 \times 10^{-4}$</td>
<td>3.537 mm</td>
<td>13805 Pa</td>
</tr>
<tr>
<td>$10\kappa_0$</td>
<td>3337</td>
<td>$1.28 \times 10^{-4}$</td>
<td>0.2232 mm</td>
<td>1619 Pa</td>
</tr>
</tbody>
</table>
Conclusions

1. We solve the Biot model by using a multiphysics reformulation. A coupled and a decoupled algorithms are developed. The approximation accuracy of the algorithms are examined under different physics parameters.

2. We compare our test results with the existing work to validate our model (and data). Then, we simulate the swelling of an injured brain.

3. The effects of each key parameter are studied: 1. the values of $E$ and $\nu$ will not affect the max ICP (but will affect the max tissue displacement); 2. the permeability has the greatest impact on the max ICP and the max deformation (low permeability will make brain edema more severe). 3. Increasing $E$, $\nu$, and $\kappa$ will make the swelling develop much faster.
Acknowledgement

Thank you!

I have an open position for postdoc.