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A Mathematical Journal

"Dedicated to Professor Gaston M. N'Guérékata on the occasion of his 60th birthday".

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Preface

Issue dedicated to Professor Gaston M. N’Guérékata on the occasion of his 60th birthday

We are happy and honored to be Guest Editors of this Issue of the “Cubo A Mathematical Journal” dedicated to our friend and colleague Gaston M. N’Guérékata, professor of mathematics at Morgan State University, Baltimore, MD, USA, on the occasion of his 60th birthday.

Professor N’Guérékata earned a Ph.D. degree in mathematics at the University of Montreal, Montreal, Québec, Canada, under Professor Samuel Zaidman’s supervision and attended the University of California at Berkeley on a postdoctorate program. He is twice tenured as a Full Professor at the University of Bangui in the Central African Republic and Morgan State University in Baltimore, MD, USA (since 2003). He is a Fulbrighter, African Academic of Sciences Fellow, and a Member of the Council for African American Researchers in the Mathematical Sciences. In addition, he received numerous honors, including the “Dr. Santye Jean McIntire, II, International Award”, “the NAFEO Noble Award”, from France, USA, and the Central African Republic for his contributions to research and higher education. He was also the Claytor-Wooddard Invited Lecturer at the Mathematics Joint Meetings in San Antonio, TX, USA, in January 2006.

The research career of Professor N’Guérékata is inspiring and unique in mathematical world. He obtained his Ph.D in 1980, then went to Africa with almost no research opportunities. He held several very high positions in the government of the Central African Republic, then 15 years later he came back to the Unite States and resumed successfully with research. To our best knowledge, no other mathematician has ever done this.

Professor N’Guérékata has made important contributions in the field of evolution equations in abstract spaces. Especially, by writing his 2001 book on almost automorphic functions with values in Banach and locally convex linear spaces, and its application to differential equations in Banach and Hilbert spaces, he made a resurgence of a theory abandoned for many years since S. Bochner and W. A. Veech, and did influence on the rebirth of applications of almost automorphic functions to
evolution equations. He authored over 160 research publications, including 5 research monographs in the areas of evolution equations, harmonic analysis, and the theory of almost periodic and almost automorphic functions. He is the Chief Editor and Founder of the International Journal of Evolution Equations and the Journal of Nonlinear Evolution Equations and Applications, Associate Editor at the Applicable Analysis, and a member of the editorial boards of 12 more journals.

In selecting papers for this issue, our primary focus has been to obtain papers with theoretical and applied studies on the theory of evolution equations and nonlinear analysis. We have also several papers dealing with problems in related branches of mathematics.

Claudio Cuevas, Bruno de Andrade e Jin Liang
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Weak Solutions of Fractional Order Pettis Integral
Inclusions with Multiple Time Delay in Banach Spaces

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ABSTRACT

We study the existence of weak solutions for nonlinear integral inclusion with multiple
time delay. The main result of the paper is based on the fixed point theorem of Mönch
type and the technique of measure of weak noncompactness.

RESUMEN

Estudiamos la existencia de soluciones débiles de la inclusión integral no lineal con
retardos temporales múltiples. El resultado principal del artículo se basa en el Teorema
de Punto Fijo de tipo Mönch y la técnica de medida de la no-compacidad débil

Keywords and Phrases: Hyperbolic differential inclusion, measure of weak noncompactness,
left sided mixed Pettis integral, weak solution, Banach space.

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1 Introduction

Fractional differential equations have been of great interest recently. It is due to the development of the theory of fractional calculus itself and by application of such constructions in various fields of science and engineering such as control theory, physics, mechanics, electrochemistry, porous media, etc. There are many works discussing the solvability of nonlinear fractional differential equations and inclusions, see the monographs of Abbas et al. [2], Kilbas et al. [14], Lakshmikantham et al. [15], Podlubny [18], Tarasov [20], the papers of Agarwal et al. [3, 4, 5], Benchohra et al. [7, 8], Kilbas and Marzan [13], Salem [19], Vityuk and Golushkov [21], and the references therein.

In [12], R. W. Ibrahim and H. A. Jalab studied the existence of solutions of the following fractional integral inclusion

\[ u(t) - \sum_{i=1}^{m} b_i(t) u(t - \tau_i) \in \mathcal{I}^\alpha F(t, u(t)); \text{ if } t \in [0, T], \]  

where \( \tau_i < t \in [0, T] \), \( b_i : [0, T] \to \mathbb{R} \), \( i = 1 \ldots, n \) are continuous functions, and \( F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R}) \) is a given multivalued map.

In [1], Abbas and Benchohra considered the following fractional integral equation with delay

\[ u(x, y) = \sum_{i=1}^{m} g_i(x, y) u(x - \xi_i, y - \mu_i) + I_0^r f(x, y, u(x, y)); \text{ if } (x, y) \in \tilde{J} := [0, a] \times [0, b], \]  

\[ u(x, y) = \Phi(x, y); \text{ if } (x, y) \in \tilde{J} := [-\xi, a] \times [-\mu, b] \setminus (0, a) \times (0, b), \]  

where \( a, b > 0 \), \( \theta = (0, 0) \), \( \xi_i, \mu_i \geq 0 \); \( i = 1 \ldots, m \), \( \xi = \max_{i=1, \ldots, m} \xi_i \), \( \mu = \max_{i=1, \ldots, m} \mu_i \), \( I_0^r \) is the left-sided mixed Riemann-Liouville integral of order \( r = (r_1, r_2) \in (0, \infty) \times (0, \infty) \), \( f : J \times \mathbb{R}^n \to \mathbb{R}^n \), \( g_i : J \to \mathbb{R} \); \( i = 1 \ldots, m \) are given continuous functions, and \( \Phi : \tilde{J} \to \mathbb{R}^n \) is a given continuous function such that

\[ \Phi(x, 0) = \sum_{i=1}^{m} g_i(x, 0) \Phi(x - \xi_i, -\mu_i); x \in [0, a], \]  

and

\[ \Phi(0, y) = \sum_{i=1}^{m} g_i(0, y) \Phi(-\xi_i, y - \mu_i); y \in [0, b]. \]

Motivated by the above papers, in this paper, we consider the following fractional integral inclusion with multiple time delay:

\[ u(x, y) - \sum_{i=1}^{m} g_i(x, y) u(x - \xi_i, y - \mu_i) \in I_0^\alpha F(x, y, u(x, y)); \quad (x, y) \in J_a \times J_b. \]  

\[ u(x, y) = \Psi(x, y); \quad (x, y) \in \tilde{J} = [-\xi, a] \times [-\mu, b] \setminus (0, a) \times (0, b), \]
where $J_a = [0, a]$, $J_b = [0, b]$ for $a, b > 0$, $\theta = (0, 0)$, $\xi = \max_{i=1, \ldots, m}[\xi_i]$, $\mu = \max_{i=1, \ldots, m}[\mu_i]$, $I_0^\alpha$ is the left sided mixed Pettis integral of order $\alpha$, $\alpha = (\alpha_1, \alpha_2) \in (0, \infty) \times (0, \infty)$, $F : J_a \times J_b \times E \rightarrow P(E)$ is a multivalued map ($P(E)$ is the family of all nonempty subsets of $E$), $g_i : J_a \times J_b \rightarrow \mathbb{R}; i = 1, \ldots, m$ are given continuous functions, and $\Psi : \tilde{J} \rightarrow E$ is a given continuous function such that

$$
\Psi(0, y) = \sum_{i=1}^m g_i(0, y)\Psi(-\xi_i, y - \mu_i); y \in [0, b],
$$

and

$$
\Psi(x, 0) = \sum_{i=1}^m g_i(x, 0)\Psi(x - \xi_i, -\mu_i); x \in [0, a].
$$

$E$ is a Banach space with norm $\|\cdot\|$. Our result is based on fixed point theorem of Mönch type and the technique of measure of weak noncompactness. Let us mention that other tools like the nonlinear alternative of Leray-Schauder type, the Banach fixed point theorem and Schauder’s fixed point theorem, such have been used to analyze the above problem in the scalar case [1, 2]. The present results complement and extend those considered in the scalar case.

## 2 Preliminaries

In this section, we introduce the notation, definitions, and preliminary facts that will be used in the remainder of this survey paper. Let $\mathbb{R}$ denote the real line and let $J_a = [0, a]$ and $J_b = [0, b]$ be two closed and bounded intervals in $\mathbb{R}$ for some real numbers $a > 0$ and $b > 0$. Throughout the paper, $E$ is a Banach space with norm $\|\cdot\|$ and dual $E^*$. Also $(E, w) = (E, \sigma(E, E^*))$ denotes the space $E$ with its weak topology. We take $C(J_a \times J_b, E)$ to be the Banach space of continuous functions $u : J_a \times J_b \rightarrow E$, with the usual supremum norm

$$
\|u\|_{\infty} = \sup\{\|u(x, y)\|, (x, y) \in J_a \times J_b\}.
$$

**Definition 2.1.** [7] The function $x : J_a \times J_b \rightarrow E$ is said to be Pettis integrable on $J_a \times J_b$ if and only if there is an element $x_{1 \times J} \in E$ corresponding to each $1 \times J \subset J_a \times J_b$ ($1$ and $J$ are measurable), such that $\varphi(x_{1 \times J}) = \int_1^J \int_J \varphi(x(s, t))\,ds\,dt$ for all $\varphi \in E^*$ where the integral on the right is assumed to exist in the sense of Lebesgue (by definition, $x_{1 \times J} = \int_1^J \int_J x(s, t)\,ds\,dt$).

We let $L^1(J_a \times J_b, E)$ denote the Banach space of measurable functions $u : J_a \times J_b \rightarrow E$ that are Pettis integrable, equipped with the norm

$$
\|u\|_{L^1} = \int_a^b \int_0^b \|u(x, y)\|\,dx\,dy.
$$

Let $P(E)$ is the family of all nonempty subsets of $E$.

A multivalued map $G : E \rightarrow P(E)$ has convex (closed) valued if $G(x)$ is convex (closed) for all $x \in E$. We say that $G$ is bounded on bounded sets if $G(B)$ is bounded in $E$ for each bounded set
B of E (i.e. sup_{x \in B} [sup \{\|y\| : y \in G(x)\}] < \infty). The map G is upper semicontinuous (u.s.c) on E if for each x_0 \in E, the set G(x_0) is a nonempty closed subset of E and for each open set N of E containing G(x_0) there exists an open neighborhood M of x_0 such that G[M] \subseteq N. The mapping G has a fixed point if there exists x \in E such that x \in G(x).

In what follows P_{cl}(E) = \{Y \in P(E) : Y is closed\}, P_b(E) = \{Y \in P(E) : Y is bounded\}, P_{cp}(E) = \{Y \in P(E) : Y is compact\}, and P_{cp,cv}(E) = \{Y \in P(E) : Y is compact and convex\}.

A multivalued map G : J_a \times J_b \to P_{cl}(E) is said to be measurable if for each \omega \in E the function

\[(x, y) \to d(\omega, G(x, y)) = \inf\{|\omega - v| : v \in G(x, y)|\}

is measurable. For more details on multivalued maps see the books of Aubin and Cellina [6], Deimling [10].

**Definition 2.2.** A function h : E \to E is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to weakly convergent sequence in E (i.e for any \(\{x_n\}_n \in E \) with \(x_n \to x \) in \(E, w\), \(h(x_n) \to h(x)\) in \(E, w\)).

**Definition 2.3.** A function F : Q \to P_{cp,cv}(Q) has weakly sequentially closed graph if for any sequence \(\{x_n, y_n\} \in Q \times Q\), where \(y_n \in F(x_n)\) for \(n \in \{1, 2, \ldots\}\) and where both \(x_n \to x\) in \(E, \omega\) and \(y_n \to y\) in \(E, \omega\) then \(y \in F(x)\).

**Proposition 2.4.** [7] [11] If \(x(.)\) is Pettis integrable and \(h(.)\) is a measurable and essentially bounded real-valued function, then \(x(.)h(.)\) is Pettis integrable.

**Definition 2.5.** [7] Let E be a Banach space, \(\Omega_E\) the bounded subsets of E and \(B_1\) the unit ball of E. The De Blasi measure of weak noncompactness is the map \(\beta : \Omega_E \to [0, \infty)\) defined by

\[\beta(X) = \inf\{\epsilon > 0 : \text{there exists a weakly compact subset } \Omega \text{ of } E : X \subseteq \epsilon B_1 + \Omega\}\]

**Properties:** De Blasi measure of noncompactness satisfies some properties

(a) \(A \subset B \Rightarrow \beta(A) \leq \beta(B)\),

(b) \(\beta(A) = 0 \Rightarrow A\) is relatively compact,

(c) \(\beta(A \cup B) = \max\{\beta(A), \beta(B)\}\),

(d) \(\beta(\overline{A^w}) = \beta(A)\), (\(\overline{A^w}\) denotes the weak closure of A),

(e) \(\beta(A + B) \leq \beta(A) + \beta(B)\),

(f) \(\beta(\lambda A) = |\lambda|\beta(A)\),

(g) \(\beta(\text{conv}(A)) = \beta(A)\),

(h) \(\beta(\bigcup_{|\lambda| \leq \epsilon} \lambda A) = h\beta(A)\).

The following result follows directly from the Hahn-Banach theorem.

**Proposition 2.6.** Let E be a normed space with \(x_0 \neq 0\) then there exists \(\varphi \in E^*\) with \(\|\varphi\| = 1\) and \(\varphi(x_0) = \|x_0\|\).
For a given set $V$ of functions $v : J_a \times J_b \to E$ let us denote by
$$V(x, y) = \{ v(x, y) : v \in V, (x, y) \in J_a \times J_b \}$$
and
$$V(J_a \times J_b) = \{ v(x, y) : v \in V, (x, y) \in J_a \times J_b \}.$$ 
For completeness, we recall the definition of the fractional Pettis-integral of order $\alpha > 0$.

Let $\alpha_1, \alpha_2 > 0$ and $\alpha = (\alpha_1, \alpha_2)$. For $h \in L^1(J_a \times J_b, E)$, the expression
$$\left( I_{\alpha}^0 h \right)(x, y) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1} h(s, t) ds dt$$
where the sign $\int$ denotes the Pettis integral and $\Gamma(.)$ is the Euler gamma function, is called the left sided mixed Pettis integral of order $\alpha$.

For our purpose we will need the following fixed point theorem.

**Theorem 2.7.** [10] Let $E$ be a Banach space with $Q$ a nonempty, bounded, closed, convex and equicontinuous subset of metrizable locally convex vector space $C(J, E)$ such that $0 \in Q$. Suppose that $T : Q \to P_{cl,cv}(Q)$ has weakly-sequentially closed graph. If the implication
$$V = \text{conv}(\{0\} \cup T(V)) \Rightarrow V$$
is relatively weakly compact, then the operator $T$ has a fixed point.

3 Main Results

we first define what we mean by solution of the problem (4)-(5).

**Definition 3.1.** A function $u \in C(J_a \times J_b, E)$ is said to be solution of problem (4)-(5) if there exists a function $v \in L^1(J_a \times J_b, E)$ with $v(x, y) \in F(x, y, u(x, y))$ and such that
$$u(x, y) = \sum_{i=1}^m g_i(x, y) u(x - \xi_i, y - \mu_i) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1} v(s, t) ds dt$$
and the function $u$ satisfies condition (5) on $\tilde{J}$.

For any $u \in C(J_a \times J_b, E)$, we define the set
$$S_{F,u} = \{ v \in L^1(J_a \times J_b, E), v(x, y) \in F(x, y, u(x, y)), (x, y) \in J_a \times J_b \}.$$
This is known as the set of selection function. Set
$$G = \max_{i=1,...,m} \sup_{(x,y) \in J_a \times J_b} |g_i(x, y)|.$$
We are now in the position to state and prove our existence result for the problem (4)-(5). We first list the following hypotheses.
(H1) \( F : J_a \times J_b \times E \to P_{cp, cl, cv}(E) \), has weakly sequentially closed graph.

(H2) For each \( u \in C(J_a \times J_b, E) \), there exists a measurable function \( v : J_a \times J_b \to E \) with \( v(x, y) \in F(x, y, u(x, y)) \) a.e. on \( J_a \times J_b \) and \( v \) is Pettis integrable on \( J_a \times J_b \).

(H3) There exists \( p \in L_\infty(J_a \times J_b, \mathbb{R}^+) \) such that

\[
\|F(x, y, u)\|_p = \sup\{\|v\| : v \in F(x, y, u)\} \leq p(x, y),
\]

for \( (x, y) \in J_a \times J_b \) and each \( u \in E \).

(H4) There exists a number \( R > 0 \) such that

\[
p^* a^{\alpha_1} b^{\alpha_2} \Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1) (1 - mG) < R,
\]

where \( p^* = \|p\|_\infty \).

(H5) Let \( r_0 > 0 \) be arbitrary (but fixed). For any \( \epsilon > 0 \) and for any subset \( X \subset B_{r_0} \), there exists a closed subset \( I_\epsilon \subset J_a \times J_b \) such that \( \mu(J_a \times J_b \setminus I_\epsilon) < \epsilon \) and

\[
\beta(F(T \times X)) \leq \sup_{(x, y) \in T} p(x, y) \beta(X),
\]

for each closed subset \( T \) of \( I_\epsilon \), where \( \mu \) denotes the Lebesgue measure in \( \mathbb{R}^2 \).

The main result in this paper reads as follows.

**Theorem 3.2.** Assume that assumptions (H1)-(H5) hold. If

\[
mG + \frac{p^* a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1)} < 1,
\]

then problem (4)-(5) has at least one solution on \( J \).

**Proof.** To transform problem (4)-(5) into a fixed point problem, we define a multivalued map \( \Omega : C(J, E) \to P_{cl}(C(J, E)) \) as

\[
\Omega(u) = \{ h \in C(J, E) \text{ such that } h(x, y) = \begin{cases} 
\Psi(x, y) & \text{if } (x, y) \in \tilde{J}, \\
\sum_{i=1}^m g_i(x, y) u(x - \xi_i, y - \mu_i) & \text{if } v \in S_{F, u}, \\
\gamma_1 \int_0^y (x - s)^{\alpha_1 - 1} (y - t)^{\alpha_2 - 1} v(s, t) ds dt & \text{if } (x, y) \in J_a \times J_b.
\end{cases}
\]

where \( \Psi(\cdot, \cdot) \) is the function defined by (5). Now, we prove that \( \Omega \) satisfies all the assumptions of the Theorem 2.7 and thus \( \Omega \) has a fixed point which is a solution of problem (4)-(5). \( \square \)
First notice that, for all $u \in C(J, E)$, there exists a Pettis integral $v : J_a \times J_b \rightarrow E$ such that $v(x, y) \in F(x, y, u(x, y))$ for a.e. $(x, y) \in J_a \times J_b$ (Assumption (H2)) then $\varphi(v(x, y)) \in L^1(J_a \times J_b)$ for any $\varphi \in E^\ast$. From the definition of the integral of fractional order we have

$$I^a \varphi(v(x, y)) = \int_0^y \int_0^x \frac{(x-s)^{\alpha-1}(y-t)^{\alpha-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \varphi(u(s, t))dsdt$$

exists for almost every $(x, y) \in J_a \times J_b$ and is an element from $L^1(J_a \times J_b)$, that is, for almost every $(x, y) \in J_a \times J_b$, $s \in [0, x)$, $t \in (0, y)$ the measurable function

$$\varphi \left( \frac{(x-s)^{\alpha-1}(y-t)^{\alpha-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} u(s, t) \right) = \frac{(x-s)^{\alpha-1}(y-t)^{\alpha-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \varphi(u(s, t))$$

is Lebesgue integrable, hence the function $(s, t) \rightarrow \frac{(x-s)^{\alpha-1}(y-t)^{\alpha-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} u(s, t)$ is Pettis integrable on $J_a \times J_b$, and thus the operator $\Omega$ is well defined.

Let $R > 0$ and consider the set

$$Q = \{ u \in C(J, E) : \|u\|_\infty \leq R \}
$$

and $\|u(x_2, y_2) - u(x_1, y_1)\| \leq R \sum_{i=1}^m |g_i(x_2, y_2) - g_i(x_1, y_1)|$

$$+ \frac{p^\ast}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)} \|x_2^{\alpha_1}y_2^{\alpha_2} - x_1^{\alpha_1}y_1^{\alpha_2}\|; \text{ for } (x_1, y_1), (x_2, y_2) \in J_a \times J_b$$

Clearly, the subset $Q$ is closed, bounded, convex and equicontinuous subset of a metrisable locally convex vector space $C(J, E)$. The remainder of the proof will be given in four steps.

**Step 1:** $\Omega(u)$ is convex for each $u \in Q$.

For that, let $0 < \lambda < 1$, $h_1, h_2 \in \Omega(u)$, obviously if $(x, y) \in \bar{J}$ then $\lambda h_1(x, y) + (1-\lambda)h_2(x, y) \in \Omega(u)$. Now if $(x, y) \in J_a \times J_b$, then there exists $v_1, v_2 \in S_{F,u}$ such that

$$h_i(x, y) = \sum_{i=1}^m g_i(x, y)u(x - \xi_i, y - \mu_i) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}v_i(s, t)dsdt; \quad i = 1, 2$$

Then for each $(x, y) \in J_a \times J_b$ we have

$$\lambda h_1 + (1-\lambda)h_2(x, y) = \sum_{i=1}^m g_i(x, y)u(x - \xi_i, y - \mu_i) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^y (x-s)^{\alpha_1-1}(y-t)^{\alpha_2-1}(\lambda v_1(s, t) + (1-\lambda)v_2(s, t))dsdt.$$
Since $S_{r,u}$ is convex (because $F$ has convex values), it follows that $\lambda h_1 + (1 - \lambda)h_2 \in \Omega(u)$.

**Step 2**: $\Omega$ maps $Q$ into $Q$.

To see this, take $h \in \Omega Q$. Then there exists $u \in Q$ with $h \in \Omega u$. And there exists $v : J_a \times J_b \to E$ Pettis integrable with $v(x, y) \in F(x, y, u(x, y))$

$$h(x, y) = \begin{cases} 
\Psi(x, y) & \text{if } (x, y) \in \bar{J}, \\
\sum_{i=1}^{m} g_i(x, y)u(x - \xi_i, y - \mu_i) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x - s)^{\alpha_1 - 1}(y - t)^{\alpha_2 - 1}v(s, t)dsdt & \text{if } v \in S_{r,u},
\end{cases}$$

$(x, y) \in J_a \times J_b$.

We can consider that $h(x, y) \neq 0$ and by Proposition 2.6 there exists $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi(h(x, y)) = \|h(x, y)\|$ for $(x, y) \in J_a \times J_b$, we have

$$\|h(x, y)\| = \varphi(h(x, y))$$

$$= \varphi \left( \sum_{i=1}^{m} g_i(x, y)u(x - \xi_i, y - \mu_i) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x - s)^{\alpha_1 - 1}(y - t)^{\alpha_2 - 1}v(s, t)dsdt \right)$$

$$= \varphi \left( \sum_{i=1}^{m} g_i(x, y)u(x - \xi_i, y - \mu_i) \right) + \varphi \left( \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x - s)^{\alpha_1 - 1}(y - t)^{\alpha_2 - 1}v(s, t)dsdt \right)$$

$$\leq m\text{GR} + \frac{p^*}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x - s)^{\alpha_1 - 1}(y - t)^{\alpha_2 - 1}dsdt$$

$$\leq m\text{GR} + \frac{p^* a^{\alpha_1 - 1} b^{\alpha_2}}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)} \leq R.$$

on the other hand, for $(x, y) \in \bar{J}$, we have

$$h(x, y) = \varphi(h(x, y)) \leq R.$$

Next, suppose that $(x_1, y_1), (x_2, y_2) \in J_a \times J_b$ with $x_1 < x_2$ and $y_1 < y_2$, and let $h \in \Omega u$, so $h(x_1, y_1) - h(x_2, y_2) \neq 0$. Then there exists $\varphi \in E^*$ such that

$$\|h(x_1, y_1) - h(x_2, y_2)\| = \varphi(h(x_1, y_1) - h(x_2, y_2)),$$

and $\|\varphi\| = 1$. Thus
\[ \|h(x_2, y_2) - h(x_1, y_1)\| \]

\[ = \sum_{i=1}^{m} g_i(x_2, y_2)u(x_2 - \xi_i, y_2 - \mu_i) \]
\[ + \frac{1}{\Gamma(a_1)/\Gamma(a_2)} \int_{0}^{x_2} \int_{y_1}^{y_2} (x_2 - s)^{a_1-1}(y_2 - t)^{a_2-1}v(s, t)dsdt \]
\[ - \sum_{i=1}^{m} g_i(x_1, y_1)u(x_1 - \xi_i, y_1 - \mu_i) \]
\[ + \frac{1}{\Gamma(a_1)/\Gamma(a_2)} \int_{0}^{x_2} \int_{y_1}^{y_2} (x_1 - s)^{a_1-1}(y_1 - t)^{a_2-1}v(s, t)dsdt \]
\[ = \sum_{i=1}^{m} g_i(x_2, y_2)u(x_2 - \xi_i, y_2 - \mu_i) - \sum_{i=1}^{m} g_i(x_1, y_1)u(x_1 - \xi_i, y_1 - \mu_i) \]
\[ + \frac{1}{\Gamma(a_1)/\Gamma(a_2)} \int_{0}^{x_2} \int_{y_1}^{y_2} (x_2 - s)^{a_1-1}(y_2 - t)^{a_2-1}v(s, t)dsdt \]
\[ + \frac{1}{\Gamma(a_1)/\Gamma(a_2)} \int_{0}^{x_2} \int_{y_1}^{y_2} (x_1 - s)^{a_1-1}(y_1 - t)^{a_2-1}v(s, t)dsdt \]
\[ \leq \sum_{i=1}^{m} \|g_i(x_2, y_2)u(x_2 - \xi_i, y_2 - \mu_i) - g_i(x_1, y_1)u(x_1 - \xi_i, y_1 - \mu_i)\| \]
\[ + \frac{1}{\Gamma(a_1)/\Gamma(a_2)} \int_{0}^{x_2} \int_{y_1}^{y_2} (x_2 - s)^{a_1-1}(y_2 - t)^{a_2-1}dsdt \]
\[ + \frac{1}{\Gamma(a_1)/\Gamma(a_2)} \int_{0}^{x_2} \int_{y_1}^{y_2} (x_1 - s)^{a_1-1}(y_1 - t)^{a_2-1}dsdt \]
\[ \leq R \sum_{i=1}^{m} \|g_i(x_2, y_2) - g_i(x_1, y_1)\| + \frac{p^*}{\Gamma(a_1 + 1)/\Gamma(a_2 + 1)} |x_2^{a_1}y_2^{a_2} - x_1^{a_1}y_1^{a_2}|. \]

This implies that \( h \in Q \), hence \( \Omega Q \subset Q \).

**Step 3:** \( \Omega \) has weakly sequentially closed graph.

Let \((u_n, w_n)\) be a sequence in \( Q \times Q \) with \( u_n(x, y) \to u(x, y) \) in \( E, w \) for each \((x, y) \in J_a \times J_b\), \( w_n(x, y) \to w(x, y) \) in \( E, w \) for each \((x, y) \in J_a \times J_b\) and \( w_n \in \Omega(u_n) \) for \( n \in \{1, 2, \ldots\} \). We show that \( w \in \Omega(u) \).

Since \( w_n \in \Omega(u_n) \), there exists \( v_n \in S_{F, u_n} \) such that

\[
w_n(x, y) = \sum_{i=1}^{m} g_i(x, y)u_n(x - \xi_i, y - \mu_i) + \frac{1}{\Gamma(a_1)/\Gamma(a_2)} \int_{0}^{x} \int_{y} (x - s)^{a_1-1}(y - t)^{a_2-1}v_n(s, t)dsdt.\]
We show that there exists \( v \in S_{\mathfrak{r}, u} \) such that
\[
w(x, y) = \sum_{i=1}^{m} g_i(x, y)u(x - \xi_i, y - \mu_i) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{0}^{y} \int_{0}^{x} (x - s)^{\alpha_1-1}(y - t)^{\alpha_2-1}u(s, t)dsdt.
\]

Since \( F(\cdot, \cdot, \cdot) \) has compact values, there exists a subsequence \( \nu_{n_m} \in S_{\mathfrak{r}, u_n} \) such that \( \nu_{n_m} \) is Pettis integrable and
\[
\nu_{n_m}(x, y) \in F(x, y, u_n(x, y)) \text{ a.e.} \ (x, y) \in J_a \times J_b
\]
and
\[
\nu_{n_m}(\cdot, \cdot) \rightarrow \nu(\cdot, \cdot) \text{ in } [E, w] \text{ as } m \rightarrow \infty.
\]

As \( F(x, y, \cdot) \) has weakly sequentially closed graph, \( \nu(x, y) \in F(x, y, u(x, y)) \). Then Lebesgue Dominated Convergence theorem for Pettis integral implies that
\[
\varphi(w_n(x, y)) \rightarrow \varphi \left( \sum_{i=1}^{m} g_i(x, y)u(x - \xi_i, y - \mu_i) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{0}^{y} \int_{0}^{x} (x - s)^{\alpha_1-1}(y - t)^{\alpha_2-1}u(s, t)dsdt \right)
\]
i.e. \( w_n(x, y) \rightarrow \Omega u(x, y) \) in \([E, w]\). Since this holds, for each \( (x, y) \in J_a \times J_b \), we have \( w \in \Omega u \).

**Step 4:** the implication \( \square \) holds.

Let \( V \) be a subset of \( Q \) such that \( \overline{V} = \overline{\text{conv}}(\Omega(V) \cup \{0\}) \). Obviously \( V(x, y) \subseteq \overline{\text{conv}}(\Omega(V(x, y)) \cup \{0\}), \forall (x, y) \in J \). Further, as \( V \) is bounded and equicontinuous, the function \( (x, y) \rightarrow \nu(x, y) = \beta(V(x, y)) \) is continuous on \( J \).

If \( (x, y) \in \overline{J} \) then
\[
\Omega V(x, y) = \{ \Omega u(x, y) : u \in V \} = \{ \Psi(x, y) : (x, y) \in \overline{J} \}
\]
and since \( \Psi \) is continuous on \( [-\xi, 0] \times [-\mu, 0] \), the set \( \{ \Psi(x, y) : (x, y) \in [-\xi, 0] \times [-\mu, 0] \} \subseteq E \) is compact. Now by (H3) and the properties of the measure \( \beta \), for any \( (x, y) \in J_a \times J_b \), we have
\[
\nu(x, y) \leq \beta(\Omega V(x, y) \cup \{0\})
\]
\[
\leq \beta(\overline{\text{conv}}(\Omega V(x, y)) \cup \{0\})
\]
\[
\leq \beta(\overline{\text{conv}}(\Omega V(x, y)))
\]
\[
\leq \beta(\Omega u(x, y) : u \in V)
\]
\[
\leq \beta \left\{ \sum_{i=1}^{m} g_i(x, y)u(x - \xi_i, y - \mu_i) : u \in V \right\}
\]
\[
+ \beta \left\{ \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{0}^{y} \int_{0}^{x} (x - s)^{\alpha_1-1}(y - t)^{\alpha_2-1}u(s, t)dsdt : u(x, y) \in F(x, y, u), u \in V \right\}
\]
\[
\leq \sum_{i=1}^{m} \beta \{ g_i(x, y)u(x - \xi_i, y - \mu_i) : u \in V \}
\]
\[ \left. + \frac{1}{\Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1)} \beta \left\{ \int_0^x \int_0^y (x-s)^{\alpha_1 - 1} (y-t)^{\alpha_2 - 1} v(s,t) ds dt; v(x,y) \in F(x,y,u), u \in V \right\} \right. \\
\leq \sum_{i=1}^m |g_i(x,y)| \beta(V(x,y)) \\
+ \frac{1}{\Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1)} \int_0^x \int_0^y (x-s)^{\alpha_1 - 1} (y-t)^{\alpha_2 - 1} p(s,t) \beta(V(s,t)) ds dt \\
\leq mG\|v\|_{\infty} + \frac{p^* a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1)} \|v\|_{\infty}. \]

In particular,
\[ \|v\|_{\infty} \leq \|v\|_{\infty} \left( mG + \frac{p^* a^{\alpha_1} b^{\alpha_2}}{\Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1)} \right). \]

By (8) it follows that \( \|v\|_{\infty} = 0 \), that is \( v(x,y) = \beta(V(x,y)) = 0 \) for each \( (x,y) \in J \) and then \( V \) is weakly relatively compact in \( C(J,E) \). Applying now Theorem 2.7 we conclude that \( T \) has a fixed point which is a solution of problem [4]-[5].

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References


ABSTRACT

We introduce a general Solow model on time scales and derive a nonlinear first-order dynamic equation that describes such a model. We first assume that there is neither technological development nor a change in the population. We present the Cobb–Douglas production function on time scales and use it to give the solution for the equation that describes the model. Next, we provide several applications of the generalized Solow model. Finally, we generalize our work by allowing technological development and population growth. The presented results not only unify the continuous and the discrete Solow models but also extend them to other cases “in between”, e.g., a quantum calculus version of the Solow model. Finally it is also noted that our results even generalize the classical continuous and discrete Solow models since we allow the savings rate, the depreciation factor of goods, the growth rate of the population, and the technological growth rates to be functions of time rather than taking constant values as in the classical Solow models.

Keywords and Phrases: Time scales, Solow model, dynamic equation, Cobb–Douglas production function, economics.

2010 AMS Mathematics Subject Classification: 91B64, 34C10, 39A10, 39A11, 39A12, 39A13.
Introducimos un modelo general de Solow en escalas de tiempo y derivamos una ecuación dinámica no lineal de primer orden que describe el mencionado modelo. Primero asumimos que no existe ni desarrollo tecnológico ni un cambio en la población. Presentamos la función de producción de Cobb-Douglas en escalas de tiempo y la utilizamos para entregar la solución de la ecuación que describe el modelo. Luego, mostramos varias aplicaciones del modelo generalizado de Solow. Finalmente, generalizamos nuestro trabajo permitiendo desarrollo tecnológico y crecimiento de la población. Los resultados presentados no sólo unifican los modelos de Solow continuos y discretos, sino que además se extienden a otros casos “entre medio”, es decir, una versión del cálculo cuántico del modelo de Solow. Finalmente, también se menciona que nuestro resultado también generaliza los modelos clásicos continuos y discretos, ya que permitimos tasas de ahorro, el factor de depreciación de bienes, la razón de crecimiento de la población y las razones de crecimiento tecnológico por ser funciones del tiempo más que asumiendo valores constantes como es el caso de los modelos de Solow clásico.

1 The Classical Solow Model

Modern growth theory is mainly based on the works of Solow [12] and Swan [13]. In the Solow model, it is assumed that the national income \( Y \) depends on consumption \( C \) and investment \( I \), i.e.,

\[
Y(t) = C(t) + I(t).
\]

Moreover, it is assumed that the national income is a function of the capital stock \( K \) and the product of the technological progress \( A \) and the population \( N \), i.e.,

\[
Y(t) = F(K(t), A(t)N(t)),
\]

where the production function \( F \) satisfies the following conditions:

1. \( F(\lambda K, \lambda L) = \lambda F(K, L) \) for all \( \lambda, K, L \in \mathbb{R}^+ \) (constant returns to scale);
2. \( F(K, 0) = F(0, L) = 0 \) for all \( K, L \in \mathbb{R}^+ \);
3. \( \frac{\partial F}{\partial K} > 0, \frac{\partial F}{\partial L} > 0, \frac{\partial^2 F}{\partial K^2} < 0, \frac{\partial^2 F}{\partial L^2} < 0; \)
4. \( \lim_{K \to 0^+} \frac{\partial F}{\partial K} = \lim_{L \to 0^+} \frac{\partial F}{\partial L} = +\infty, \lim_{K \to +\infty} \frac{\partial F}{\partial K} = \lim_{L \to +\infty} \frac{\partial F}{\partial L} = 0. \)

Furthermore, the change of the capital stock in a particular period does not only depend on the new investment but also on the depreciation of goods. In other words, we take for granted that

\[
K'(t) = I(t) - \delta K(t)
\]
with given initial capital stock $K(0)$, where $\delta$ is the depreciation rate of the goods. As usual, we presume that

$$S(t) = I(t) = sY(t), \quad (3)$$

where $s$ is the savings rate, and thus the savings $S$ is the portion of the national income which is not consumed. Plugging (1) and (3) into (2), we obtain the nonlinear first-order differential equation

$$K'(t) = sF(K(t), A(t)N(t)) - \delta K(t). \quad (4)$$

In this paper, we assume that the technological knowledge of the society grows exponentially with rate $r$, that is,

$$A'(t) = rA(t), \quad \text{i.e.,} \quad A(t) = e^{rt}A(0),$$

where $A(0)$ is the initial technological standing of the nation. Similarly, we suppose that the population of the nation grows with growth rate $n$ and an initial population of $N(0)$, hence

$$N'(t) = nN(t), \quad \text{i.e.,} \quad N(t) = e^{nt}N(0).$$

Both $n$ and $r$ can be positive or negative depending on which nation we are talking about. Of course, it seems quite unrealistic that $r$ is negative. Solow also took the capital stock per efficiency of labor $k$ into account, which he defined as the stationary variable

$$k(t) := \frac{K(t)}{A(t)N(t)}.$$

We use constant returns in order to define the intensive version of the production function to be

$$y(t) := \frac{Y(t)}{A(t)N(t)} = \frac{F(K(t), A(t)N(t))}{A(t)N(t)} = F(k(t), 1) =: f(k(t)).$$

This allows us now to rewrite (4) as

$$\frac{K'(t)}{A(t)N(t)} = sf(k(t)) - \delta k(t). \quad (5)$$

A simple calculation shows that

$$\frac{k'(t)}{k(t)} = \frac{K'(t)}{K(t)} - r - n.$$ 

Therefore (5) turns into

$$k'(t) = sf(k(t)) - (\delta + n + r)k(t). \quad (6)$$

Table 1 summarizes all the variables with their meanings that appear in the Solow model. For a more detailed discussion of stability and qualitative analysis of the Solow model, the reader might consult [9]. The discrete analogue of Solow’s model is discussed in [11], featuring some results similar to those in the continuous Solow model. So far this dynamic process was regarded either as solely continuous or solely discrete. In this paper, we generalize these two theories in such a way that the continuous and the discrete versions of the Solow model are only special cases of
Table 1: Explanation of Variables

<table>
<thead>
<tr>
<th>Variable</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>national income</td>
</tr>
<tr>
<td>I</td>
<td>induced investment</td>
</tr>
<tr>
<td>C</td>
<td>consumption</td>
</tr>
<tr>
<td>S</td>
<td>Savings</td>
</tr>
<tr>
<td>s</td>
<td>savings rate</td>
</tr>
<tr>
<td>K</td>
<td>capital stock</td>
</tr>
<tr>
<td>K₀</td>
<td>initial capital stock</td>
</tr>
<tr>
<td>δ</td>
<td>depreciation factor of goods</td>
</tr>
<tr>
<td>A</td>
<td>level of technological knowledge</td>
</tr>
<tr>
<td>A₀</td>
<td>initial technological progress level</td>
</tr>
<tr>
<td>r</td>
<td>technological growth rate</td>
</tr>
<tr>
<td>N</td>
<td>level of population</td>
</tr>
<tr>
<td>N₀</td>
<td>initial population</td>
</tr>
<tr>
<td>n</td>
<td>growth rate of population</td>
</tr>
<tr>
<td>k</td>
<td>capital stock per efficiency unit of labor</td>
</tr>
</tbody>
</table>

our generalized Solow model. We do this using the time scales theory, an area of mathematics which was originally introduced by Stefan Hilger in his PhD thesis [10]. The two books [7, 8] by Bohner and Peterson offer an introduction with applications to time scales calculus along with some advanced topics. Applications of time scales calculus can be found in many areas, also in economics. Tisdell and Zaidi [14] in particular already generalized some economic topics, and also Bohner et al. discussed multiplier-accelerator models and utility functions on time scales in [5, 6].

The set up of this paper is as follows. In Section 2, we give a brief introduction to the time scales theory. In Section 3, we present the Solow model on time scales and derive the nonlinear first-order dynamic equation that describes this model. In Section 4, we define the generalized Cobb–Douglas production function on time scales and provide examples for various time scales. Furthermore, we state a theorem that gives the solution of the nonlinear first-order dynamic equation, and we also provide examples for several time scales. We finally state a result that addresses asymptotic stability of the solution. Section 5 is used to show some important properties of the production function, called the Inada conditions. Finally, in Section 6, we point out how our model can be extended, assuming the presence of technological and population growth (or decay), i.e., by assuming \( n \neq 0 \) as well as \( r \neq 0 \). We present the Cobb–Douglas production function for
this more general case and also derive the equilibrium solution for this extended model.

It is also noted that our results even generalize the classical continuous and discrete Solow models since we allow the savings rate, the depreciation factor of goods, the growth rate of the population, and the technological growth rates to be functions of time rather than considering constant values as in the classical Solow models.

2 Time Scales Preliminaries

In this section, we introduce some elements of time scales calculus. For a more rigorous time scales introduction, we refer the reader to [7, 8].

Let $T$ be a time scale, i.e., a nonempty closed subset of $\mathbb{R}$. For $t \in T$, the forward jump operator $\sigma : T \to T$ is defined by

$$\sigma(t) := \inf \{s \in T : s > t\},$$

while the backward jump operator $\rho : T \to T$ is defined by

$$\rho(t) := \sup \{s \in T : s < t\}.$$

In this definition, we set $\inf \emptyset = \sup T$ (i.e., $\sigma(t) = t$ if $T$ has maximum $t$) and $\sup \emptyset = \inf T$ (i.e., $\rho(t) = t$ if $T$ has minimum $t$). If $\sigma(t) > t$, $\sigma(t) = t$, $\rho(t) < t$, and $\rho(t) = t$, then $t$ is called right-scattered, right-dense, left-scattered and left-dense, respectively. The graininess function $\mu : T \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

We also need the set $T^\kappa$ which is defined in the following way: If $T$ has a left-scattered maximum $m$, then $T^\kappa = T - \{m\}$. Else, $T^\kappa = T$.

Now let $f : T \to \mathbb{R}$ be a function. If $t \in T^\kappa$, then $f^\Delta(t)$ is defined as the number (provided that it exists) such that for every $\epsilon > 0$, there exists a neighborhood $U$ of $t$ (i.e., $U = (t - \delta, t + \delta) \cap T$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s)) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

We call this number $f^\Delta(t)$ the delta derivative of $f$ at $t$. Moreover, $f$ is called rd-continuous provided it is continuous at right-dense points in $T$ and its left sided limits exist (finite) at left-dense points in $T$. The function $f^\sigma : T \to \mathbb{R}$ is defined by $f^\sigma = f \circ \sigma$. In our calculations, we use the so-called “simple useful formula”

$$f^\sigma = f + \mu f^\Delta.$$

We denote the set of rd-continuous functions by $C_{rd} = C_{rd}(T) = C_{rd}(T, \mathbb{R})$. Next, $f$ is said to be regressive given that

$$1 + \mu(t)f(t) \neq 0 \quad \text{for all} \quad t \in T.$$
holds. The set of all regressive and rd-continuous functions is denoted by \( \mathcal{R} = \mathcal{R}(T) = \mathcal{R}(T, \mathbb{R}) \).

We also define the set \( \mathcal{R}^+ \) of all positively regressive elements by

\[
\mathcal{R}^+ = \mathcal{R}^+(T, \mathbb{R}) = \{ f \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in T \}.
\]

Let now \( p, q \in \mathcal{R} \). We define the “circle plus” addition \( \oplus \) on \( \mathcal{R} \) by

\[
(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t) \quad \text{for all } t \in T
\]

and the “circle minus” subtraction \( \ominus \) on \( \mathcal{R} \) by

\[
(p \ominus q)(t) := \frac{p(t) - q(t)}{1 + \mu(t)q(t)} \quad \text{for all } t \in T.
\]

We put

\[
\mathcal{R}(\alpha) := \begin{cases} \mathcal{R} & \text{if } \alpha \in \mathbb{N}, \\ \mathcal{R}^+ & \text{if } \alpha \in \mathbb{R} \setminus \mathbb{N}. \end{cases}
\]

For \( \alpha \in \mathbb{R} \) and \( p \in \mathcal{R}(\alpha) \), we define

\[
(\alpha \circ p)(t) := \alpha p(t) \int_0^t (1 + \mu(t)p(t)h)^{\alpha-1}dh. \tag{7}
\]

The time scales exponential function \( e_p(\cdot, t_0) \) is defined for \( p \in \mathcal{R} \) and \( t_0 \in T \) as the unique solution of the initial value problem

\[
y^\Delta = p(t)y, \quad y(t_0) = 1 \quad \text{on } T.
\]

We have

\[
e_p(\cdot, t_0)e_q(\cdot, t_0) = e_{p \oplus q}(\cdot, t_0) \quad \text{and} \quad \frac{e_p(\cdot, t_0)}{e_q(\cdot, t_0)} = e_{p \ominus q}(\cdot, t_0).
\]

If \( \alpha \in \mathbb{R} \) and \( p \in \mathcal{R}(\alpha) \), then

\[
e_{\alpha \circ p} = e_p^\alpha
\]

(see [8, Theorem 2.44]). Let \( \alpha \in \mathbb{R} \setminus \{1\} \). We say that

\[
x^\Delta = \left[ q \odot \left( \frac{1}{\alpha-1} \odot (gx^{\alpha-1}) \right) \right] x
\]

(see [8, Section 2.6]) is a Bernoulli equation on time scales.

### 3 Solow Model on Time Scales

Assume that \( F \) and \( f \) are production functions as defined in Section [1]. We now introduce the generalized Solow model on an arbitrary time scale:

\[
\begin{align*}
Y(t) & = F(K(t), A(t)N(t)), \\
K^\Delta(t) & = I(t) - \delta(t)K(t), \\
I(t) & = s(t)Y(t), \\
A^\Delta(t) & = r(t)A(t), \\
N^\Delta(t) & = n(t)N(t),
\end{align*}
\]

(9)
where we require

$$\delta(t) > 0 \quad \text{and} \quad s(t) > 0 \quad \text{for all} \quad t \in \mathbb{T}$$

(10)

and

$$n, r \in \mathbb{R}.$$  

(11)

The economical meanings of $\delta, s, r,$ and $n$ are the same as described in Table 1. If $(K, Y, A, N, I)$ solves (9), then

$$K^\Delta(t) = s(t)Y(t) - \delta(t)K(t) = s(t)F(K(t), A(t)N(t)) - \delta(t)K(t).$$

(12)

Define

$$k(t) := \frac{K(t)}{A(t)N(t)} \quad \text{and} \quad y(t) := \frac{Y(t)}{A(t)N(t)},$$

(13)

which are regarded as the capital stock per efficiency unit of labor and the production per efficiency unit of labor, respectively. By (12) and (13), we have

$$K^\Delta(t)A(t)N(t) = s(t)f(k(t)) - \delta(t)k(t).$$

(14)

**Theorem 3.1.** Assume (9), (10), and (11). If $k$ is defined as in (13), then

$$k^\Delta(t) = \frac{s(t)}{(1 + \mu(t)r(t))(1 + \mu(t)n(t))}{f(k(t))} - \left(\frac{\delta(t) + n(t)}{(1 + \mu(t)r(t))(1 + \mu(t)n(t))} + \frac{r(t)}{1 + \mu(t)r(t)}\right)k(t).$$

(15)

**Proof.** The time scales quotient rule [7, Theorem 1.20 (v)] provides

$$k^\Delta = \left(\frac{K}{AN}\right)^\Delta = \frac{K^\Delta AN - K(AN^\Delta + A^\Delta N^\sigma)}{AN^\sigma N^\sigma} = \frac{K^\Delta}{A^\Delta N^\sigma} - \frac{KN^\Delta}{AN^\sigma N^\sigma} - \frac{K\Delta}{AN^\sigma}$$

\[\begin{align*}
&= \frac{AN(1 + \mu r)(1 + \mu n)}{(1 + \mu r)(1 + \mu n)} - \frac{n}{(1 + \mu r)(1 + \mu n)}k - \frac{r}{1 + \mu r}k,
&= \frac{s}{(1 + \mu r)(1 + \mu n)}f(k) - \left(\frac{\delta + n}{(1 + \mu r)(1 + \mu n)} + \frac{r}{1 + \mu r}\right)k,
\end{align*}\]

i.e., (15) holds.

**Example 3.2.** If $\mathbb{T} = \mathbb{R},$ then $\sigma(t) = t$ and $\mu(t) = 0$ for all $t \in \mathbb{T}.$ Thus (15) can be rewritten as

$$k'(t) = s(t)f(k(t)) - (\delta(t) + n(t) + r(t))k(t),$$

which reduces to (6) provided $\delta, s, n,$ and $r$ are constants.
Theorem 3.3. Assume (10) and (11). Then (15) holds if and only if
\[ k^\Delta(t) = s(t)f(k(t)) - \delta(t)k(t) - (n \oplus r)(t)k(\sigma(t)). \] (16)

Proof. Suppose \( k \) solves (16). Then we use the “simple useful formula” to obtain
\[ k^\Delta = s(f \circ k) - \delta k - (n \oplus r)k^\sigma = s(f \circ k) - \delta k - (n + r + \mu r)(k + \mu k^\Delta). \]

Hence
\[ (1 + \mu n)(1 + \mu r)k^\Delta = s(f \circ k) - (\delta + n + r + \mu nr)k. \]

Dividing by \( (1 + \mu n)(1 + \mu r) \) yields (15). If \( k \) solves (15), then (16) follows by reversing the above steps.

Theorem 3.4. Assume (10) and (11). If (16) holds, then
\[ (1 + \mu(t)n(t))(1 + \mu(t)r(t))k^\sigma(t) = \mu(t)s(t)f(k(t)) + (1 - \mu(t)\delta(t))k(t). \] (17)

If (17) holds and \( \mu(t) \neq 0 \), then (16) holds.

Proof. Suppose \( k \) solves (16). Then we multiply (16) by \( \mu(t) \) and use the simple useful formula to obtain
\[ k(\sigma(t)) - k(t) = \mu(t)k^\Delta(t) \]
\[ = \mu(t)s(t)f(k(t)) - \mu(t)\delta(t)k(t) - \mu(t)(n \oplus r)(t)k(\sigma(t)). \]

Hence
\[ (1 + \mu(t)(n \oplus r)(t))k(\sigma(t)) = \mu(t)s(t)f(k(t)) + (1 - \mu(t)\delta(t))k(t), \]

which results in (17). If (17) holds at \( t \in T \) such that \( \mu(t) \neq 0 \), then the above steps can be reversed.

Example 3.5. If \( T = \mathbb{Z} \), then \( \sigma(t) = t + 1 \) and \( \mu(t) = 1 \) for all \( t \in T \). Thus (17) can be rewritten as
\[ (1 + n(t))(1 + r(t))k(t + 1) = s(t)f(k(t)) + (1 - \delta(t))k(t). \]

This equation can be found in [11].

Example 3.6. If \( T = h\mathbb{Z} \) with \( h > 0 \), then \( \sigma(t) = t + h \) and \( \mu(t) = h \) for all \( t \in T \). Thus (17) can be rewritten as
\[ (1 + hn(t))(1 + hr(t))k(t + h) = hs(t)f(k(t)) + (1 - h\delta(t))k(t). \]

Example 3.7. If \( T = q^{\mathbb{N}_0} \) with \( q > 1 \), then \( \sigma(t) = qt \) and \( \mu(t) = (q - 1)t \) for all \( t \in T \). Thus (17) can be rewritten as
\[ (1 + (q - 1)tn(t))(1 + (q - 1)tr(t))k(qt) = (q - 1)ts(t)f(k(t)) + (1 - (q - 1)t\delta(t))k(t). \]
4 Analysis of the Basic Solow Model

In this section, we assume that \( \text{(10)} \) holds and that there is no technological development and no population change, i.e., \( n = r = 0 \). Then \( \text{(16)} \) simplifies to
\[
k^A(t) = s(t)f(k(t)) - \delta(t)k(t). \tag{18}
\]

Let
\[
0 < \alpha < 1, \quad w(t) = \left( \frac{1}{\alpha - 1} \odot \frac{\delta g}{s} \right)(t), \quad \text{and} \quad g(t) = (1 - \alpha)s(t). \tag{19}
\]
If
\[
\tilde{f}(x) := \delta(t) + \left( w \odot \left( \frac{1}{\alpha - 1} \odot (gx^{\alpha - 1}) \right) \right)(t) \quad \text{is independent of} \quad t \in \mathbb{T}, \tag{20}
\]
then we define the generalized Cobb–Douglas production function on time scales by
\[
f(x) = x\tilde{f}(x). \tag{21}
\]

**Theorem 4.1.** Let \( t \in \mathbb{T} \). If \( \mu(t) = 0 \), then
\[
\delta(t) + \left( w \odot \left( \frac{1}{\alpha - 1} \odot (gx^{\alpha - 1}) \right) \right)(t) = x^{\alpha - 1}. \tag{22}
\]

**Proof.** Assume \( \mu(t) = 0 \). Then at \( t \), we have
\[
\delta(t) + \left( w \odot \left( \frac{1}{\alpha - 1} \odot (gx^{\alpha - 1}) \right) \right)(t) = \frac{\delta(t) + w(t) - \frac{g(t)x^{\alpha - 1}}{\alpha - 1}}{s(t)} = \frac{\delta(t) + \frac{\delta(t)g(t)}{(\alpha - 1)s(t)} - \frac{g(t)x^{\alpha - 1}}{\alpha - 1}}{s(t)} = \frac{\delta(t) - \delta(t) + s(t)x^{\alpha - 1}}{s(t)} = x^{\alpha - 1},
\]
which shows \( \text{(22)} \). \( \square \)

**Example 4.2.** If \( \mathbb{T} = \mathbb{R} \), then \( \tilde{f}(x) = x^{\alpha - 1} \), and thus \( \text{(20)} \) holds. Hence the Cobb–Douglas production function is defined and equals
\[
f(x) = x\tilde{f}(x) = xx^{\alpha - 1} = x^\alpha.
\]

**Theorem 4.3.** Let \( t \in \mathbb{T} \). If \( \mu(t) > 0 \), then
\[
\delta(t) + \left( w \odot \left( \frac{1}{\alpha - 1} \odot (gx^{\alpha - 1}) \right) \right)(t) = \frac{1}{\mu(t)} \left( \delta(t)\mu(t) - 1 + \left( \frac{1 + (1 - \alpha)\mu(t)s(t)x^{\alpha - 1}}{1 + (1 - \alpha)\mu(t}\delta(t)} \right) \right)^{-1}. \tag{23}
\]
Proof. Assume \( \mu(t) > 0 \). Then at \( t \), we have

\[
\frac{1}{\alpha - 1} \odot (g(x^{\alpha - 1})) = \frac{1}{\alpha - 1} g(x^{\alpha - 1}) \int_0^1 \left(1 + \mu g(x^{\alpha - 1}) h\right)^{\frac{1}{\alpha - 1} - 1} dh
= \frac{(1 + \mu g(x^{\alpha - 1}))^{\frac{1}{\alpha - 1} - 1}}{\mu},
\]

\[
w = \frac{1}{\alpha - 1} \odot \frac{\delta s}{s} = \frac{1}{\alpha - 1} \odot \delta(1 - \alpha)
= \frac{1}{\alpha - 1} \delta(1 - \alpha) \int_0^1 \left(1 + \mu \delta(1 - \alpha) h\right)^{\frac{1}{\alpha - 1} - 1} dh
= \frac{(1 + \mu \delta(1 - \alpha))^{\frac{1}{\alpha - 1} - 1}}{\mu},
\]

and hence

\[
w \odot \left(\frac{1}{\alpha - 1} \odot (g(x^{\alpha - 1}))\right) = \frac{w - \frac{(1 + \mu g(x^{\alpha - 1}))^{\frac{1}{\alpha - 1} - 1}}{\mu}}{1 + \mu \frac{(1 + \mu g(x^{\alpha - 1}))^{\frac{1}{\alpha - 1} - 1}}{\mu}}
= \frac{w - \frac{1 + \mu \delta(1 - \alpha))^{\frac{1}{\alpha - 1} - 1}}{\mu}}{(1 + \mu g(x^{\alpha - 1}))^{\frac{1}{\alpha - 1}}}
= \frac{1}{\mu} \left\{ -1 + \left(\frac{1 + \mu \delta(1 - \alpha)}{1 + \mu(1 - \alpha) s x^{\alpha - 1}}\right)^{\frac{1}{\alpha - 1}} \right\},
\]

which shows (23).

Theorem 4.4. Let \( \mu(t) > 0 \) for all \( t \in T \). Assume (10) and suppose

\[
\tilde{s} := s(t) \mu(t) \quad \text{and} \quad \tilde{\delta} := \delta(t) \mu(t) \quad \text{are independent of} \quad t \in T.
\]

Then (20) holds and the Cobb–Douglas production function is defined and equals

\[
f(x) = \frac{x}{\tilde{s}} \left\{ \tilde{\delta} - 1 + \left(\frac{1 + (1 - \alpha) \tilde{s} x^{\alpha - 1}}{1 + (1 - \alpha) \tilde{\delta}}\right)^{\frac{1}{\alpha - 1}} \right\}.
\]
Proof. Using Proposition 4.3, we see that
\[
\begin{align*}
\delta(t) + \left( w \odot \left( \frac{1}{\alpha - 1} \odot (g x^{\alpha - 1}) \right) \right)(t) \\
&\quad \overset{(23)}{=} \frac{1}{\mu(t)s(t)} \left\{ \delta(t)\mu(t) - 1 + \left( \frac{1 + (1 - \alpha)\mu(t)s(t)x^{\alpha - 1}}{1 + (1 - \alpha)\mu(t)\delta(t)} \right)^{1/\alpha} \right\} \\
&\quad = \frac{1}{s} \left\{ \delta - 1 + \left( \frac{1 + (1 - \alpha)\delta x^{\alpha - 1}}{1 + (1 - \alpha)\delta} \right)^{1/\alpha} \right\}
\end{align*}
\]
is independent of \( t \) and therefore equals \( \tilde{f}(x) \). By (21), \( f(x) = x\tilde{f}(x) \).

Example 4.5. If \( T = hZ \) with \( h > 0 \) and \( \delta, s \) are constants and satisfy (10), then Proposition 4.4 gives us that (20) is satisfied and that the Cobb–Douglas production function is defined and equals (note that (24) is satisfied in this case with \( \tilde{s} = sh \) and \( \tilde{\delta} = \delta h \))
\[
f(x) = x \left\{ \delta h - 1 + \left( \frac{1 + (1 - \alpha)h x^{\alpha - 1}}{1 + (1 - \alpha)h \delta} \right)^{1/\alpha} \right\}.
\]

Example 4.6. If \( T = Z \) and \( \delta, s \) are constants and satisfy (10), then Example 4.5 gives us that (20) is satisfied and that the Cobb–Douglas production function, i.e., the discrete version of the classical Cobb–Douglas production function, is defined and equals
\[
f(x) = x \left\{ \delta - 1 + \left( \frac{1 + (1 - \alpha)\delta x^{\alpha - 1}}{1 + (1 - \alpha)\delta} \right)^{1/\alpha} \right\}.
\]

Example 4.7. If \( T = q^{\mathbb{N}_0} \) with \( q > 1 \) and \( \delta, s \) satisfy (10) and (24), i.e.,
\[
\tilde{s} := (q - 1)ts(t) \quad \text{and} \quad \tilde{\delta} := (q - 1)t\delta(t) \quad \text{are independent of} \quad t \in T,
\]
then Proposition 4.4 gives us that (20) is satisfied and that the Cobb–Douglas production function is defined and equals
\[
f(x) = x \left\{ \tilde{\delta} - 1 + \left( \frac{1 + (1 - \alpha)\tilde{s} x^{\alpha - 1}}{1 + (1 - \alpha)\tilde{\delta}} \right)^{1/\alpha} \right\}.
\]

Using (21), we rewrite equation (18) in the form (8), i.e., as a Bernoulli equation on time scales.

Theorem 4.8. Assume (10) and (20) and let \( t \) be defined by (21). Then (18) holds if and only if
\[
k^\alpha(t) = \left\{ w \odot \left( \frac{1}{\alpha - 1} \odot (g k^{\alpha - 1}) \right) \right\}(t)k(t).
\]
Example 4.9. If $T = \mathbb{R}$, then $w = -\delta$, and equation (26) is
\[ k'(t) = (sk^{\alpha - 1}(t) - \delta)k(t). \]

Hence equation (26) is indeed a generalized form of the continuous Solow model with the Cobb–Douglas production function.

With the generalized Cobb–Douglas function, we can find the solution of the Solow model (18) on time scales.

Theorem 4.10. Assume (10) and
\[ \kappa := \frac{s(t)}{\delta(t)} \text{ is independent of } t \in T \quad (27) \]
and define $p \in \mathcal{R}$ by
\[ p(t) := (1 - \alpha)\delta(t) \text{ for all } t \in T. \quad (28) \]

Then the solution of (26) with initial condition $k(t_0) = k_0 > 0$, where $t_0 \in T$, is given by
\[ k(t) = \left\{ \kappa + \frac{k_0^{\alpha - 1} - \kappa}{e_p(t,t_0)} \right\}^{\frac{1}{1-\alpha}} \text{ for all } t \in T, \quad (29) \]
provided the quantity in curly braces in (29) is always positive.

Proof. Suppose $k$ solves (26) such that $k(t_0) = k_0$. Define $\tilde{x} := k^{\alpha - 1}$. By [8, Theorem 2.37], we have
\[ \frac{\tilde{x}^\Delta}{\tilde{x}} = (\alpha - 1) \odot \frac{k^\Delta}{k} \]
\[ = (\alpha - 1) \odot \left\{ w \odot \left[ \frac{1}{\alpha - 1} \odot (gk^{\alpha - 1}) \right] \right\} \]
\[ = [(\alpha - 1) \odot w] \odot (gk^{\alpha - 1}) \]
\[ = (\delta(1 - \alpha)) \odot (gk^{\alpha - 1}), \]
so
\[ \tilde{x}^\Delta = (p \odot (g\tilde{x})) \tilde{x}, \]
which shows that $\tilde{x}$ solves the logistic equation on time scales (see [1] and [8, Section 2.4]). Define $y := 1/\tilde{x}$. Then
\[ y^\Delta = \left( \frac{1}{\tilde{x}} \right)^\Delta = -\frac{x^\Delta}{xx^\sigma} = -(p \odot (g\tilde{x}))y^\sigma = \frac{g\tilde{x} - p}{1 + \mu g\tilde{x}}y^\sigma \]
and hence
\[ (1 + \mu g\tilde{x})y^\Delta = g\tilde{x}y^\sigma - py^\sigma, \]
i.e., using the “simple useful formula”
\[ y^\Delta + (y^\sigma - y)g\tilde{x} = g\tilde{x}y^\sigma - py^\sigma, \]
i.e.,
\[ y^\Delta = -py^\sigma + g. \]  
(30)

Using \( g = (1 - \alpha)s = (1 - \alpha)\delta \kappa = \kappa p \)
and the variation of constants formula [7, Theorem 2.74], the solution of (30) is given by

\[ y(t) = y_0 e^{\text{⊖}p(t, t_0)} + \int_{t_0}^{t} g(\tau)e^{\text{⊖}p(t, \tau)} \Delta \tau \]

From the substitutions we performed, we have that \( y_0 = k_0^{1-\alpha} \) as well as \( k(t) = \frac{1}{y(t)^{\frac{1}{\alpha}}} \), which shows (29). Conversely, \( k \) given by (29) is easily seen to be a solution of (26).

Using Theorem 4.10 we obtain the asymptotic stability of the unique equilibrium point of (26).

**Theorem 4.11.** Assume (10) and (27). If
\[ \int_{t_0}^{\infty} \delta(t) \Delta t = \infty, \]  
(31)
then any solution \( k \) of (26) in the form (29) satisfies
\[ \lim_{t \to \infty} k(t) = \kappa^{\frac{1}{1-\alpha}} =: \kappa, \]
and \( \kappa \) is the unique equilibrium point of (26).

**Proof.** We have \( p(t) > 0 \) for all \( t \in T \), where \( p \) is defined in Theorem 4.10. Hence \( p \in \mathbb{R}^+ \). Thus, by [4, Remark 2], we have
\[ e_p(t, t_0) \geq 1 + \int_{t_0}^{t} p(\tau) \Delta \tau = 1 + (1 - \alpha) \int_{t_0}^{t} \delta(\tau) \Delta \tau \]
for all \( t \geq t_0 \).

Therefore, using (31),
\[ \lim_{t \to \infty} e_p(t, t_0) = \infty, \]
and thus
\[ \lim_{t \to \infty} k(t) = \left\{ \kappa + \frac{k_0^{1-\alpha} - \kappa}{\lim_{t \to \infty} e_p(t, t_0)} \right\}^{\frac{1}{1-\alpha}} = \kappa^{\frac{1}{1-\alpha}} = \kappa. \]
Now we show that $\kappa$ is the unique nontrivial equilibrium point of (26): A point $\kappa$ is a nontrivial equilibrium point of (26) if and only if

$$w \ominus \left( \frac{1}{\alpha - 1} \otimes (g_\kappa^{\alpha-1}) \right) = 0,$$

which holds if and only if (use the definition of $w$, (27), and the properties of $\ominus$)

$$\frac{1}{\alpha - 1} \otimes \frac{g}{\kappa} = \frac{1}{\alpha - 1} \otimes (g_\kappa^{\alpha-1}),$$

which is true if and only if (use the properties of $\otimes$ and the definition of $g$)

$$\frac{1}{\kappa} = \kappa^{\alpha-1},$$

i.e., $\kappa = \kappa$. This completes the proof.

**Example 4.12.** Let $T = \mathbb{R}$ and assume (10) and (27). Then (29) reads

$$k(t) = \left\{ \kappa + \frac{k_0^{1-\alpha} - \kappa}{e^{(1-\alpha) \int_0^t \delta(\tau) d\tau}} \right\}^{\frac{1}{\alpha - 1}}.$$

If, in addition, $t_0 = 0$ and $\delta$ is constant (this implies that $s$ is constant as well), then

$$k(t) = \left\{ \kappa + \frac{k_0^{1-\alpha} - \kappa}{e^{(1-\alpha) \delta t}} \right\}^{\frac{1}{\alpha - 1}}.$$

**Example 4.13.** Let $T = h\mathbb{Z}$ with $h > 0$ and assume (10) and (27). Then (29) reads

$$k(t) = \left\{ \kappa + \frac{k_0^{1-\alpha} - \kappa}{\prod_{i=0}^{t/h - 1} (1 + (1-\alpha)h\delta(ih))} \right\}^{\frac{1}{\alpha - 1}}.$$

If, in addition, $t_0 = 0$ and $\delta$ is constant (this implies that $s$ is constant as well), then

$$k(t) = \left\{ \kappa + \frac{k_0^{1-\alpha} - \kappa}{(1 + (1-\alpha)h\delta)^{t/h}} \right\}^{\frac{1}{\alpha - 1}}.$$

**Example 4.14.** Let $T = q^{N_0}$ with $q > 1$ and assume (10) and (27). Then (29) reads

$$k(t) = \left\{ \kappa + \frac{k_0^{1-\alpha} - \kappa}{\prod_{i=\log_q t_0}^{\log_q t - 1} (1 + (q-1)q^i(1-\alpha)\delta(q^i))} \right\}^{\frac{1}{\alpha - 1}}.$$
If, in addition, \( t_0 = 1 \) and \( \tilde{\delta} := (q - 1)t_0 \) is constant (this implies that \( (q - 1)t_0 \) is constant as well), then

\[
k(t) = \left\{ \kappa + \frac{k_0^{1 - \alpha} - \kappa}{(1 - \alpha)\tilde{\delta}} \right\}^{\frac{1}{\alpha}}.
\]

5 Properties of the Production Function

In this section, we show that our Cobb–Douglas production function \( f \) given in (21) satisfies the time scales Inada conditions (see [2, 3])

\[
\begin{align*}
(32) \\
&f(x) > 0, \quad \tilde{f}'(x) < 0, \quad f''(x) < 0 \quad \text{for all} \quad x > 0, \\
&\tilde{f}(x) > \zeta(t) \geq 0, \quad f'(x) > \zeta(t) \geq 0 \quad \text{for all} \quad x > 0 \quad \text{and all} \quad t \in T, \\
&\lim_{x \to 0^+} \tilde{f}(x) = \lim_{x \to 0^+} f'(x) = \infty, \\
&\lim_{x \to \infty} \tilde{f}(x) = \lim_{x \to \infty} f'(x) = \zeta(t) \geq 0 \quad \text{for all} \quad t \in T,
\end{align*}
\]

where \( \zeta : T \to \mathbb{R} \) is defined in the following lemma.

Lemma 5.1. Assume (10) and define

\[ \zeta(t) := \frac{1}{s(t)} \left\{ \delta(t) + \left( \frac{1}{\alpha - 1} \circ (1 - \alpha) \delta(t) \right) (t) \right\}. \]

Then \( \zeta(t) \geq 0 \).

Proof. If \( \mu(t) = 0 \), then \( \zeta(t) = 0 \). If \( \mu(t) > 0 \), then, as in the proof of Proposition 4.3 we have

\[ \left( \frac{1}{\alpha - 1} \circ (1 - \alpha) \delta(t) \right) (t) = \frac{(1 + (1 - \alpha)\mu(t)\delta(t))^{-\frac{1}{\alpha - 1}} - 1}{\mu(t)}. \]

Now using the well-known Bernoulli inequality, we obtain

\[ (1 + (1 - \alpha)\mu(t)\delta(t))^{-\frac{1}{\alpha - 1}} \geq 1 + \frac{1}{\alpha - 1}(1 - \alpha)\mu(t)\delta(t) = 1 - \mu(t)\delta(t). \]

This proves the claim. \( \Box \)

Theorem 5.2. Assume (10) and (20) and define the Cobb–Douglas production function by (21).

If there exists \( t \in T \) such that \( \mu(t) = 0 \), then the Cobb–Douglas production function satisfies the Inada conditions (32).

Proof. By Theorem 4.1 and (20),

\[ f(x) = x^\alpha \quad \text{and} \quad \tilde{f}(x) = x^{\alpha - 1}. \]  

(33)

Clearly, \( f \) given by (33) satisfies the Inada conditions (32). \( \Box \)
Theorem 5.3. Assume (10) and (20) and define the Cobb–Douglas production function by (21).
If there exists $t \in T$ such that $\mu(t) > 0$, then the Cobb–Douglas production function satisfies the Inada conditions (32).

Proof. By Theorem 4.3 and (20), we have
\[ \tilde{f}(x) = \frac{1}{\mu(t)s(t)} \left\{ \mu(t)\delta(t) - 1 + \left( \frac{1 + (1 - \alpha)\mu(t)s(t)x^{\alpha-1}}{1 + (1 - \alpha)\mu(t)\delta(t)} \right)^{\frac{1}{1-\alpha}} \right\}. \] (34)

In order to check that the Inada conditions (32) are satisfied, we use the following notation:
\[ \begin{align*}
\tilde{\delta} &:= \mu(t)\delta(t), \\
\kappa &:= \frac{s(t)}{\delta(t)}, \\
\tilde{\alpha} &:= (1 - \alpha)\tilde{\delta}, \\
\zeta &:= \frac{1}{\tilde{\delta} \kappa} \left\{ \tilde{\delta} - 1 + (1 + \tilde{\alpha})^{\frac{1}{1-\alpha}} \right\}, \\
z(x) &:= \frac{1 + \tilde{\alpha}xx^{\alpha-1}}{1 + \tilde{\alpha}}. 
\end{align*} \] (35)

Using (35), we rewrite (34) as
\[ \tilde{f}(x) = \frac{1}{\tilde{\delta} \kappa} \left\{ \tilde{\delta} - 1 + (z(x))^{\frac{1}{1-\alpha}} \right\}. \] (36)

We obviously have
\[ z(x) > \frac{1}{1 + \tilde{\alpha}} \quad \text{for all} \quad x > 0. \] (37)

Using (37) in (36), we find
\[ \tilde{f}(x) > \frac{1}{\tilde{\delta} \kappa} \left\{ \tilde{\delta} - 1 + (1 + \tilde{\alpha})^{\frac{1}{1-\alpha}} \right\} = \zeta \quad \text{for all} \quad x > 0. \]

By Lemma 5.1 we also have $\zeta \geq 0$, and hence $\tilde{f}(x) > 0$ for all $x > 0$ so that $f(x) = xf'(x) > 0$ for all $x > 0$. Next, note that
\[ z'(x) = \frac{\tilde{\alpha}(\alpha - 1)\kappa}{1 + \tilde{\alpha}}x^{\alpha-2}. \] (38)

Using (38) in (36), we find
\[ \begin{align*}
\tilde{f}'(x) &= \frac{1}{\tilde{\delta} \kappa (1 - \alpha)} \left( z(x) \right)^{\frac{1}{1-\alpha} - 1} z'(x) \\
&= \frac{1}{\tilde{\delta} \kappa (1 - \alpha)} \left( z(x) \right)^{\frac{\alpha}{1-\alpha}} \frac{\tilde{\alpha}(\alpha - 1)\kappa x^{\alpha-2}}{1 + \tilde{\alpha}} \\
&= -\frac{\tilde{\alpha}x^{\alpha-2}}{(1 + \tilde{\alpha})\delta} \left( z(x) \right)^{\frac{\alpha}{1-\alpha}} < 0 \quad \text{for all} \quad x > 0.
\end{align*} \]
Using this, (36), (35), (37), and the product rule for \( f(x) = x\tilde{f}(x) \), we get

\[
f'(x) = \tilde{f}(x) + x\tilde{f}'(x)
= \frac{1}{\delta k} \left\{ \tilde{\delta} - 1 + (z(x))^{\frac{\alpha}{\alpha - \tilde{\alpha}}} \right\}
- \frac{\tilde{\alpha} x^{\alpha - 1} \kappa}{(1 + \tilde{\alpha}) \delta k} (z(x))^{\frac{\alpha}{\alpha - \tilde{\alpha}}}
= \frac{1}{\delta k} \left\{ \tilde{\delta} - 1 + \frac{1}{1 + \tilde{\alpha}} (z(x))^{\frac{\alpha}{\alpha - \tilde{\alpha}}} \right\}
- \left( z(x) - \frac{1}{1 + \tilde{\alpha}} \right) \frac{1}{\delta k} (z(x))^{\frac{\alpha}{\alpha - \tilde{\alpha}}}
> \frac{1}{\delta k} \left\{ \tilde{\delta} - 1 + \frac{1}{1 + \tilde{\alpha}} (1 + \tilde{\alpha})^{\frac{\alpha}{\alpha - \tilde{\alpha}}} \right\}
= \frac{1}{\delta k} \left\{ \tilde{\delta} - 1 + (1 + \tilde{\alpha})^{\frac{\alpha}{\alpha - \tilde{\alpha}}} \right\} = \zeta,
\]

i.e.,

\[
f'(x) = \frac{1}{\delta k} \left\{ \tilde{\delta} - 1 + \frac{1}{1 + \tilde{\alpha}} (z(x))^{\frac{\alpha}{\alpha - \tilde{\alpha}}} \right\} > \zeta \quad \text{for all } x > 0. \tag{39}
\]

Also, since

\[
\lim_{x \to 0^+} z(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} z(x) = \frac{1}{1 + \tilde{\alpha}},
\]

we find from (36) and (39) that

\[
\lim_{x \to 0^+} \tilde{f}(x) = \lim_{x \to \infty} f'(x) = \infty
\]

and

\[
\lim_{x \to \infty} \tilde{f}(x) = \lim_{x \to \infty} f'(x) = \zeta \geq 0.
\]

Finally, using (39) and (38), we obtain

\[
f''(x) = \frac{1}{\delta k (1 + \tilde{\alpha})} \frac{\alpha}{1 - \alpha} (z(x))^{\frac{\alpha}{\alpha - \tilde{\alpha}}} - 1 z'(x)
= \frac{1}{\delta k (1 + \tilde{\alpha})} \frac{\alpha}{1 - \alpha} \tilde{\alpha} (\alpha - 1) \kappa x^{\alpha - 2} (z(x))^{\frac{\alpha - 1}{\alpha - \tilde{\alpha}}}
= -\frac{\alpha \tilde{\alpha}}{\delta (1 + \tilde{\alpha})^2} x^{\alpha - 2} (z(x))^{\frac{\alpha - 1}{\alpha - \tilde{\alpha}}} < 0 \quad \text{for all } x > 0.
\]

This shows that all conditions in (32) are satisfied. \( \square \)

### 6 General Solow Model on Time Scales

Let us now assume (10) and (11) so that we allow now that the technology develops exponentially and/or the population increases exponentially. All results from Section 4 and Section 5 still hold true when we replace \( s \) and \( \delta \) in (18) by

\[
\frac{s}{1 + \mu (n \oplus r)} \quad \text{and} \quad \frac{\delta + (n \oplus r)}{1 + \mu (n \oplus r)},
\]
respectively, as (13) in this case results in (15). This means that \( w \) and \( g \) from (19) are replaced by

\[
 w(t) = \left( \frac{1}{\alpha - 1} \odot \left( \delta + (n \oplus r) g \right) \right)(t) \quad \text{and} \quad g(t) = \frac{[1 - \alpha]s(t)}{1 + \mu(t)(n \oplus r)(t)},
\]

respectively. Next, \( \tilde{f}(x) \) in (20) is replaced by

\[
 \frac{\delta(t) + (n \oplus r)(t) + (1 + \mu(t)(n \oplus r)(t))(w \odot (\frac{1}{\alpha - 1} \odot (gx^{\alpha - 1})))}{s(t)}.
\]

Moreover, \( \kappa \) and \( p \) in (27) and (28) are replaced by

\[
 \frac{s(t)}{\delta(t) + (n \oplus r)(t)} \quad \text{and} \quad (1 - \alpha) \frac{\delta(t) + (n \oplus r)(t)}{1 + \mu(t)(n \oplus r)(t)},
\]

respectively. Finally, condition (31) turns into

\[
 \int_{t_0}^{\infty} \frac{\delta(t) + (n \oplus r)(t)}{1 + \mu(t)(n \oplus r)(t)} \Delta t = \infty.
\]

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References


On Fokker-Planck and linearized Korteweg-de Vries type equations with complex spatial variables

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ABSTRACT

In the present work, we construct solutions to a Fokker-Planck type equation with real time variable and complex spatial variable, and prove some properties. The equations are obtained from the complexification of the spatial variable by two different methods. Firstly, one complexifies the spatial variable in the corresponding convolution integral in the solution, by replacing the usual sum of variables (translation) by an exponential product (rotation). Secondly, one complexifies the spatial variable directly in the corresponding evolution equation and then one searches for analytic solutions. These methods are also applied to a linear evolution equation related to the Korteweg-de Vries equation.

RESUMEN

En este trabajo construimos soluciones de una ecuación tipo Fokker-Planck con variable de tiempo real y variable espacial compleja. Las ecuaciones se obtienen de la complejización de la variable espacial por dos métodos diferentes. Primero, se complejiza la variable espacial en la integral de convolución respectiva en la solución reemplazando la suma usual de las variables (traslaciones) por un producto de exponenciales (rotación). Luego, se complejiza la variable espacial directamente en la respectiva la ecuación de evolución y se busca por las soluciones analíticas. Estos métodos también se aplican a la ecuación de evolución lineal relacionada a la ecuación Korteweg-de Vries.

Keywords and Phrases: Fokker-Planck equation, Korteweg-de Vries equation, complex convolution integrals, complex spatial variables.

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1 Dedicated to Professor Gaston N’Guerekata on the occasion of his 60th birthday.
1 Introduction

Let us consider the following initial value problem:
\[
\begin{array}{l}
\frac{\partial u}{\partial t}(t, x) = \alpha(t) \frac{\partial^2 u}{\partial x^2}(t, x) + \beta(t) xu(t, x), \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\
u(0, x) = f(x), \ x \in \mathbb{R},
\end{array}
\]
(1)

where \(\alpha \in C([0, +\infty), \mathbb{R}_+)\) and \(\beta \in C([0, +\infty), \mathbb{R})\). Using the exponential operator method, it is shown in [6] that
\[
u(t, x) = e^{a(t)+d(t)} \left[ c(t) d(t) + b(t) + x \right] 2 \pi c(t) \\
\int_{-\infty}^{\infty} e^{-(u+b(t)+2c(t)d(t))^2/(4c(t))} f(x-u) du,
\]
(2)
is a solution of (1) provided that the integral in (2) converges, with \(a(t), b(t), c(t), d(t)\) depending on \(\alpha(t), \beta(t)\), and given in [6] (54) as follows:
\[
c(t) = \int_0^t \alpha(u) du, \quad d(t) = \int_0^t \beta(u) du,
\]
(3)
\[
b(t) = -2 \int_0^t \beta(u) \int_0^u \alpha(s) ds du,
\]
\[
a(t) = 2 \int_0^t \beta(u) \left\{ \int_0^u \beta(s) \int_0^s \alpha(v) dv ds \right\} du.
\]

Note that by the assumptions on \(\alpha, \beta\), the functions \(a, b, c, d\) are differentiable for every \(t > 0\), and that \(c(t) > 0\), for all \(t > 0\). For \(\beta(t) \equiv 0\) and \(\alpha(t) \equiv C\) (\(C:\) constant), we recapture the initial value problem for the classical heat equation. For \(\beta(t) \neq 0\), \(\alpha(t) \neq 0\), the main equation in (1) is known as a Fokker-Planck type equation.

In the second part of this article, we’ll devote our attention to the "linearized" Korteweg-de Vries equation with real time variable and complex spatial variable. Indeed, let us consider the well-known Korteweg-de Vries equation
\[
\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} = \frac{\partial^3 u}{\partial x^3},
\]
(4)
where \(\alpha \in \mathbb{R}\) (see, e.g., Widder [7]), and the related linear problem
\[
\begin{array}{l}
\frac{\partial u}{\partial t}(t, x) = \frac{\partial^3 u}{\partial x^3}(t, x), \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\
\lim_{t \to 0} u(t, x) = f(x), \ x \in \mathbb{R}.
\end{array}
\]
(5)

For problem (5), the following is known to hold.

**Theorem 1.1.** (7, Theorem 4]) Let \(f: \mathbb{R} \to \mathbb{R}\) be continuous and of bounded variation, such that it satisfies the conditions:

(i) The integral
\[
F(s) = \int_{-\infty}^\infty e^{-sx} f(x) dx
\]
converges absolutely for \( s \in \mathbb{C} \) with \( \text{Re}(s) = c \);

(ii) Let \( \int_{-\infty}^{\infty} |f(c + i\tau)|d\tau < \infty \).

Setting \( u(t, x) := \int_{-\infty}^{\infty} K(x - y, t)f(y)dy = \int_{-\infty}^{\infty} K(u, t)f(x - u)du, \) (6)

where \( K(u, t) := \frac{1}{\pi} \int_{0}^{\infty} \cos(ut - \tau^3) d\tau = \frac{1}{(3t)^{1/3}} \text{Ai} \left( \frac{-u}{(3t)^{1/3}} \right), \) (7)

the function \( u(t) \) satisfies (5).

Above,

\[ \text{Ai}(u) := \frac{1}{\pi} \int_{0}^{\infty} \cos(ut + \tau^3/3) d\tau \]

is also called the Airy function. Moreover, it is well-known (see, e.g., Widder [7, (5.1)]) that

\[ \frac{\partial K}{\partial t}(x, t) = \frac{\partial^3}{\partial x^3} K(x, t), \quad t > 0, \ x \in \mathbb{R}. \] (8)

It is natural to ask what happens if in the above equations we complexify the spatial variable and keep the time variable real? We shall proceed as follows. The complexification of the spatial variable in the above mentioned equations is made by two different methods which produce different equations: first, one complexifies the spatial variable in the corresponding formula for the solutions in (2) and (5), respectively, by replacing in the integral the usual sum of variables (translation) by an exponential product (rotation) and looking for solutions in a disk \( D_R \) of radius \( R > 1 \). This method yields solutions that satisfy differential equations similar to (2) and (5). Secondly, one directly complexifies the spatial variable in the corresponding evolution equations, and then one searches for analytic and non-analytic solutions for the resulting equation. The topic was already developed in detail for complex heat and Laplace equations in [1, 2], for complex wave and telegraph equations in [3] and for complex Schrödinger type equations in [4].

2 Generalized heat type equations with complex spatial variable

Let \( R \geq 1 \) and let us now consider the open disk \( D_R = \{ z \in \mathbb{C}; |z| < R \} \) and \( A(D_R) = \{ f; \mathcal{D}_R \to \mathbb{C}; \) \( f \) is analytic on \( D_R \), continuous on \( \overline{D}_R \} \), endowed with the uniform norm \( \| f \|_R = \sup \{ |f(z)|; z \in \overline{D}_R \} \).

It is well-known that \( (A(D_R), \| \cdot \|_R) \) is a Banach space. If \( f \in A(D_R) \), then we have \( f(z) = \sum_{k=0}^{\infty} a_k z^k \), for all \( z \in D_R \). Finally, \( \omega_{\delta}(f; \delta)_{\mathcal{D}_R} \) denotes the modulus of continuity, \( \omega_{\delta}(f; \delta)_{\mathcal{D}_R} = \sup \{ |f(u) - f(v)|; \| u - v \| \leq \delta, u, v \in \overline{D}_R \}. \)
Theorem 2.1. Let $\theta$ be any $\theta > 0$, we will first complexify the solution in (2) as follows. For $f \in A(D_R)$ and $t > 0$, let us replace $x$ and the translation $x - u$ in (2) by $z$ and the rotation $ze^{-iu}$, respectively, and consider the complex integral

$$G_t(f)(z) = \frac{e^{a(t) + d(t)|c(t)d(t) + b(t) + z|}}{2\sqrt{\pi c(t)}} \int_{-\infty}^{\infty} e^{-[u+b(t)+2c(t)d(t)]^2/(4c(t))} f(ze^{-iu}) du, \ z \in \overline{D}_R. \quad (9)$$

The first goal of this section is to prove some properties of the complex integral (9).

Theorem 2.1. Let $R > 1$ and $f \in A(D_R)$.

(i) For all $t > 0$, $G_t(f) \in A(D_R)$, namely $G_t(f)$ is continuous on $\overline{D}_R$, is analytic in $D_R$, and the following holds:

$$G_t(f)(z) = e^{\Phi(t,z)} \sum_{k=0}^{\infty} a_k d_k(t) z^k, \ z \in D_R, \quad (10)$$

where

$$\Phi(t, z) = a(t) + d(t)|c(t)d(t) + b(t) + z|$$

and, for all $k \geq 0$,

$$d_k(t) := e^{-k^2c(t) + ikg(t)} \quad g(t) := b(t) + 2c(t)d(t).$$

(ii) Setting

$$W_t(f)(z) := e^{-\Phi(t,z)} G_t(f)(z) = \frac{1}{2\sqrt{\pi c(t)}} \int_{-\infty}^{\infty} f(ze^{-iu}) e^{-[u+g(t)]^2/(4c(t))} du,$$

the following estimate holds:

$$|W_t(f)(z) - f(z)| \leq (R + 1) \left[ 1 + \frac{2}{\sqrt{\pi}} + \frac{|g(t)|}{2\sqrt{c(t)}} \right] \omega_1(f; \sqrt{c(t)}) \theta_{D_R},$$

for all $z \in \overline{D}_R, \ t > 0$.

(iii) The operator $W_t$ is contractive, that is, $\|W_t(f)\|_{\overline{D}_R} \leq \|f\|_{\overline{D}_R}$, for all $t > 0, \ f \in A(D_R)$.

(iv) Let

$$U(t, \varphi) := e^{a(t)d(t)|c(t)d(t) + b(t) + \varphi|} \sum_{k=0}^{\infty} a_k d_k(t) z^k,$$

for every $z \neq 0, \ z = re^{i\varphi}$ such that $r \in (0, R), \ \varphi \in [0, 2\pi]$. Then, $U(t, \varphi)$ satisfies

$$\frac{\partial U}{\partial t} = a(t) \frac{\partial^2 U}{\partial \varphi^2} + \beta(t) \varphi U(t, \varphi), \quad (11)$$

for $(t, z) \in \mathbb{R}_+ \times D_R \setminus \{0\}, \ z = re^{i\varphi}, \ r \in (0, R), \ and$

$$U(0, \varphi) = f(re^{i\varphi}), \ \varphi \in [0, 2\pi], \ 0 < r \leq R, \ f \in A(D_R). \quad (12)$$
Proof. (i) For fixed $z \in D_R$, we have

$$ f(ze^{-iu}) = \sum_{k=0}^{\infty} a_k e^{-iku} z^k. \quad (13) $$

Since $|a_k e^{-iku}| = |a_k|$, for all $u \in \mathbb{R}$, and since $\sum_{k=0}^{\infty} a_k z^k$ is absolutely convergent, it follows that $\sum_{k=0}^{\infty} a_k e^{-iku} z^k$ is uniformly convergent with respect to $u \in \mathbb{R}$. Thus, on account of (13), we can integrate in $[0]$ term by term. This yields

$$ G_t(f)(z) = e^{\Phi(z)} \sum_{k=0}^{\infty} a_k \left[ \frac{1}{2 \sqrt{\pi c(t)}} \int_{-\infty}^{+\infty} e^{-iku} \cdot e^{-[u+g(t)]^2/(4c(t))} du \right] z^k $$

$$ = e^{\Phi(z)} \sum_{k=0}^{\infty} a_k \left[ \frac{1}{2 \sqrt{\pi c(t)}} \int_{-\infty}^{+\infty} (\cos[kv - g(t)] - i \sin[kv - g(t)]) e^{-v^2/(4c(t))} dv \right] z^k $$

$$ := e^{\Phi(z)}[I_1 - iI_2]. $$

Since $\sin(kv)e^{-v^2/(4c(t))}$ is odd as function of $v$, we have

$$ I_1 = \sum_{k=0}^{\infty} a_k \left[ \cos[kg(t)] \frac{1}{2 \sqrt{\pi c(t)}} \int_{-\infty}^{+\infty} \cos(kv)e^{-v^2/(4c(t))} dv \right] z^k $$

$$ = \sum_{k=0}^{\infty} a_k [\cos[kg(t)] \cdot e^{-k^2c(t)}] z^k $$

and

$$ I_2 = -\sum_{k=0}^{\infty} a_k [\sin[kg(t)] e^{-k^2c(t)}] z^k. $$

Hence, these calculations give the formula (10) and prove the analyticity of $G_t(f)(z)$, as function of $z \in D_R$. To prove the continuity in $D_R$, it suffices to prove the continuity in $z$, of the function

$$ H_t(f)(z) := \frac{1}{2 \sqrt{\pi c(t)}} \int_{-\infty}^{+\infty} e^{-[u+g(t)]^2/(4c(t))} f(ze^{-iu}) du. $$

To this end, let $z_0, z_n \in D_R$ be such that $\lim_{n \to \infty} z_n = z_0$. We get

$$ |H_t(f)(z_n) - H_t(f)(z_0)| \leq \frac{1}{2 \sqrt{\pi c(t)}} \int_{-\infty}^{+\infty} |f(z_n e^{-iu}) - f(z_0 e^{-iu})| e^{-[u+g(t)]^2/(4c(t))} du $$

$$ \leq \frac{1}{2 \sqrt{\pi c(t)}} \int_{-\infty}^{+\infty} \omega_1(f; |z_n e^{-iu} - z_0 e^{-iu}|) e^{-[u+g(t)]^2/(4c(t))} du $$

$$ = \frac{1}{2 \sqrt{\pi c(t)}} \int_{-\infty}^{+\infty} \omega_1(f; |z_n - z_0|) e^{-[u+g(t)]^2/(4c(t))} du $$

$$ = \omega_1(f; |z_n - z_0|) e^{-[u+g(t)]^2/(4c(t))}. $$
Passing to the limit as \( n \to \infty \), the continuity of \( H_t(f)(z) \) at \( z_0 \in \overline{D_R} \) is a consequence of (13) since \( f \) is continuous on \( \overline{D_R} \).

(ii) First, note that we can also write

\[
W_t(f)(z) = \frac{1}{2 \sqrt{\pi c(t)}} \int_{-\infty}^{+\infty} e^{-v^2/(4c(t))} f(ze^{-i(v-g(t))}) dv.
\]

A simple calculation gives

\[
|W_t(f)(z) - f(z)| \leq \frac{1}{2 \sqrt{\pi c}} \int_{-\infty}^{+\infty} |f(ze^{-i(u-g)}) - f(z)e^{-u^2/(4c)}| du \tag{15}
\]

\[
\leq \frac{R + 1}{2 \sqrt{\pi}} \int_{-\infty}^{+\infty} \omega_1(f; |1 - e^{-i(u-g)|}) e^{-u^2/(4c)} du
\]

\[
= \frac{R + 1}{2 \sqrt{\pi}} \int_{-\infty}^{+\infty} \omega_1(f; 2|\sin \frac{u - g}{2}) e^{-u^2/(4c)} du
\]

\[
\leq \frac{R + 1}{2 \sqrt{\pi}} \int_{-\infty}^{+\infty} \omega_1(f; |u - g|) e^{-u^2/(4c)} du
\]

\[
\leq (R + 1) \left[ \omega_1(f; \sqrt{c}) + \frac{\omega_1(f; \sqrt{c})}{\sqrt{2} \sqrt{\pi}} \int_{0}^{\infty} 2u e^{-u^2/(4c)} du \right]
\]

\[
+ (R + 1) \left[ |A| \cdot \frac{\omega_1(f; \sqrt{c})}{\sqrt{2} \sqrt{\pi}} \int_{0}^{\infty} e^{-u^2/(4c)} du \right].
\]

Since \( \int_{0}^{\infty} 2u e^{-u^2/(4c(t))} du = 4c(t) \), we infer

\[
|W_t(f)(z) - f(z)| \leq \omega_1(f; \sqrt{c}) \left[ 1 + \frac{2}{\sqrt{\pi}} + \frac{|g|}{2 \sqrt{c}} \right] (R + 1)
\]

This proves (ii).

(iii) Since \( \frac{1}{2 \sqrt{\pi c}} \int_{-\infty}^{+\infty} e^{-u^2/(4c)} du = 1 \), we also have

\[
|W_t(f)(z)| \leq \frac{1}{2 \sqrt{\pi c}} \int_{-\infty}^{+\infty} f(ze^{-i(u-g)})|e^{-u^2/(4c)} du \leq \|f\|_{\overline{D_R}}, \quad z \in \overline{D_R},
\]

which proves the claim.

(iv) Let \( f \in A(D_R) \), and \( z \in D_R, z = re^{i\varphi}, 0 < r < R \). Set

\[
B(t, \varphi) := a(t) + d(t)c(t) + b(t) + \varphi, \quad A_k(t) := -k^2 c(t) + ik\varphi(t);
\]

by (i), we have \( d_k(t) = e^{A_k(t)} \), and we can write

\[
U(t, \varphi) = e^{B(t, \varphi)} \sum_{k=0}^{\infty} a_k e^{A_k(t)} r^k e^{ki\varphi}.
\]
Consequently, since the series representation (10) for $G_t(f)(z)$ is uniformly convergent in any compact disk included in $D_R$, it follows that $U(t, \varphi)$ can be differentiated term by term with respect to $t$ and $\varphi$. Therefore, simple calculations show that $U$, given by (10), satisfies. We emphasize again that in (10) we must take $z \neq 0$ simply because $z = 0$ has no polar representation, that is, $z = 0$ cannot be represented as function of $\varphi$. Finally, it is also easy to check that $U(t, \varphi)$ satisfies (12) since $a(0) = b(0) = c(0) = d(0) = 0$ on account of (3). This completes the proof of the theorem.

Remark 2.2. In Theorem 2.1-(ii), we have $\lim_{t \to 0} c(t) = 0$. Therefore, there exists a sufficiently small $\delta_0 > 0$ such that $0 < c(t) < 1$, for all $t \in [0, \delta_0]$. This implies that

$$|g(t)/\sqrt{c(t)}| \leq |b(t)/c(t)| + 2|d(t)|,$$

for all $t \in (0, \delta_0)$. Exploiting (3) once more again it is easy to show that $|b(t)/c(t)| \leq 2\beta_0 t$, for all $t \in (0, \delta_0)$, where $\beta_0 = \|\beta\|_{[0, \delta_0]}$. This together with the inequality (17) yields $\lim_{t \to 0} g(t)/\sqrt{c(t)} = 0$. As a consequence, cf. the estimate of Theorem 2.1-(ii), it also follows that $\lim_{t \to 0} W_t(f)(z) = f(z)$, for all $z \in D_R$.

In what follows, the system (11) is complexified, by replacing $x \in \mathbb{R}$ with $z \in \mathbb{C}$ directly in the equations. More precisely, our goal is to study the following initial value problem

$$\left\{ \begin{array}{l}
\frac{\partial u}{\partial t}(t,z) = \alpha(t)\frac{\partial^2 u}{\partial z^2}(t,z) + \beta(t)zu(t,z), \\
u(0,z) = f(z).
\end{array} \right.
$$

We will first consider the case when $f$ is analytic. Our first goal is to search for analytic solutions $u(t,z)$, as functions of $z$, for any $t > 0$. First, we need some basic notations. For $r > 0$, define the strip

$$S_r = \{ z = x + iy \in \mathbb{C}; x \in \mathbb{R}, |y| \leq r \}
$$

and

$$A(S_r) = \{ f : S_r \to \mathbb{C}; f \text{ is analytic in } S_r \},
$$

(i.e., $f$ is analytic in a domain that contains $S_r$). Next, let $M_r$ be the set of all $f \in A(S_r)$ such that there exists $g \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ with the property that $|f'(z)| \leq g(|z|)$, as $|z|$ goes to infinity. Finally, let

$$u(t,z) := \frac{e^{\Phi(t,z)}}{2\sqrt{\pi c(t)}} \int_{-\infty}^{\infty} f(z - \xi)e^{-(\xi + g(t))\xi^2/(4c(t^2))} \, d\xi, \quad (t,z) \in \mathbb{R}_+ \times S_r,
$$

where $c(t)$, $\Phi(t,z)$, and $g(t)$ are defined in the statement of Theorem 2.1-(i).

Theorem 2.3. For each $f \in M_r$ we have $u = u(t, \cdot) \in A(S_r)$, for any $t > 0$. Moreover, $u(t,z)$ solves (12) for $(t,z) \in \mathbb{R}_+ \times S_r$. 
Proof. If \( f \in \mathcal{M}_r \), there exists \( M > 0 \) such that

\[
\sup |f'(z)|; z \in S_r = \|f'\|_{S_r} \leq M.
\]

By the Mean Value Theorem in complex analysis, \( f \) is uniformly continuous in \( S_r \), and that

\[
|f(z - \xi)| \leq |f(0)| + \|f'\|_{S_r} \cdot |z - \xi| \leq |f(0)| + M|z - \xi|.
\]

The latter implies that the integral \( I(t, z) := \int_{-\infty}^{\infty} f(z - \xi) e^{-\frac{(\xi + g(t))^2}{4c(t)}} d\xi \) exists and is absolutely convergent in \( \mathbb{C} \), and that \( I(t, z) \) is differentiable with respect to any \( z \in S_r \), with

\[
\partial_z I(t, z) = \int_{-\infty}^{\infty} f'(z - \xi) e^{-\frac{(\xi + g(t))^2}{4c(t)}} d\xi.
\]

The analitycity of \( \Phi(t, z) \) with respect to \( z \in S_r \) implies that \( u = u(t, \cdot) \) also belongs to \( A(S_r) \), for any \( t > 0 \). Analogous calculations to those performed in (15) yield

\[
\begin{align*}
|I(t, z) - f(z)| &\leq \omega_1(f; |g(t)|)_{S_r} + \omega_1(f; \sqrt{c(t)})_{S_r} + \frac{1}{2 \sqrt{\pi c(t)}} \int_{-\infty}^{\infty} \left( |u| (c(t))^{-1/2} + 1 \right) e^{-u^2/(4c(t))} du \\
&= \omega_1(f; |g(t)|)_{S_r} + \left( 1 + \frac{2}{\sqrt{\pi}} \right) \omega_1(f; \sqrt{c(t)})_{S_r}.
\end{align*}
\]

Taking now into account the Remark 2.2 and the uniform continuity of \( f \) on \( S_r \), it easily follows that \( \lim_{t \downarrow 0} I(t, z) = f(z) \), for all \( z \in S_r \). This together with the fact that \( \Phi(0, z) = 1 \) yields \( u(0, z) = \lim_{t \downarrow 0} u(t, z) = f(z) \), for all \( z \in S_r \), i.e., \( u \) satisfies the initial condition of (15). It remains to show that \( u \) also solves the main equation of (15). To this end, define

\[
F(t, z) := \frac{\partial u}{\partial t}(t, z) - \alpha(t) \frac{\partial^2 u}{\partial z^2}(t, z) - \beta(t) u(t, z),
\]

for all \( (t, z) \in \mathbb{R}_+ \times S_r \), and recall that \( u(0, z) = f(z) \), \( z \in S_r \). For each \( t > 0 \), \( F(t, \cdot) \) is analytic in \( S_r \). Taking now \( z = x \in \mathbb{R} \) in all the equations of (15), we can now apply known theory to deduce that \( u(t, z) = u(t, x) \) also solves (1). Hence, \( F(t, x) = 0 \), for all \( (t, x) \in \mathbb{R}_+ \times \mathbb{R} \). The identity theorem for holomorphic functions (in a domain that contains \( S_r \)) implies that we must also have \( F(t, z) = 0 \), \( (t, z) \in \mathbb{R}_+ \times S_r \). This finishes the proof of the theorem. \( \square \)

3 Linearized Korteweg-de Vries type equations

For \( R > 1 \), let us define the open disk

\[
D_R := \{ z \in \mathbb{C} : |z| < R \}.
\]

Next we endow the local convex space

\[
A^*(D_R) := \{ f : D_R \to \mathbb{C} : f \text{ is analytic on } D_R \},
\]
with the countable family of seminorms
\[ \|f\|_n := \sup \{|f(z)| : z \in \mathbb{D}_{R_n}, R_n \nearrow R, R_n \geq 1, \] 
and metric
\[ d(f, g) := \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\|f - g\|_n}{1 + \|f - g\|_n}. \]
Then \( A^*(\mathbb{D}_R) \) is a Fréchet space.

Let \( f \in A^*(\mathbb{D}_R) \) such that \( f(z) = \sum_{k=0}^{\infty} a_k z^k \), for all \( z \in \mathbb{D}_R \). We consider the integral operator
\[ \mathcal{T}_K(t)(f)(z) := \int_{-\infty}^{\infty} K(u, t)f(ze^{-iu})du, \quad z \in \mathbb{D}_R, \] (19)
with \( K(u, t) \) given by (7). Evidently, since \( K(-u, t) \neq K(u, t) \), we can naturally introduce another complex integral by
\[ \tilde{\mathcal{T}}_K(t)(f)(z) := \int_{-\infty}^{\infty} K(-u, t)f(ze^{iu})du, \quad z \in \mathbb{D}_R. \] (20)

The first goal of this section is to prove some properties for (19) and (20).

**Theorem 3.1.** Let \( R > 1, f \in A^*(\mathbb{D}_R) \), that is, \( f(z) = \sum_{k=0}^{\infty} a_k z^k \), for all \( z \in \mathbb{D}_R \). Let \( \mathcal{T}_K(\cdot)(f) \) and \( \tilde{\mathcal{T}}_K(\cdot)(f) \) be as in (19) and (20), respectively.

(i) For all \( t \geq 0 \), as functions of \( z \), we have \( \mathcal{T}_K(\cdot)(f)(z) \in A^*(\mathbb{D}_R), \tilde{\mathcal{T}}_K(\cdot)(f)(z) \in A^*(\mathbb{D}_R) \), and there hold
\[ \mathcal{T}_K(t)(f)(z) = \sum_{k=0}^{\infty} a_k d_k(t)z^k \quad \text{and} \quad \tilde{\mathcal{T}}_K(t)(f)(z) = \sum_{k=0}^{\infty} a_k b_k(t)z^k, \quad z \in \mathbb{D}_R, \] (21)
where for all \( k \geq 0 \),
\[ d_k(t) = e^{ikt^3} \quad \text{and} \quad b_k(t) = e^{-ikt^3}. \]
Moreover,
\[ \mathcal{T}_K(0)(f) = \tilde{\mathcal{T}}_K(0)(f) = f. \]

(ii) For all \( z \in \mathbb{D}_R \) with \( 1 \leq r < R \) and \( t \in \mathbb{R}_+ \), the following estimate holds:
\[ |\mathcal{T}_K(t)(f)(z) - f(z)| \leq \frac{1}{2} \sum_{k=0}^{\infty} |a_k|k^6r^k + |t| \sum_{k=0}^{\infty} |a_k|k^3r^k, \]
where
\[ \sum_{k=0}^{\infty} |a_k|k^6r^k < \infty, \quad \text{since} \quad f^{(6)} \in A^*(\mathbb{D}_R). \]
(iii) For all \( z \in D_r \) with \( 1 \leq r < R \) and \( t, s \in \mathbb{R}_+ \), there holds:

\[
|T_K(t)(f)(z) - T_K(s)(f)(z)| \leq 2 \sum_{k=0}^{\infty} |a_k|k^3r^k|t - s|
\]

and

\[
|\tilde{T}_K(t)(f)(z) - \tilde{T}_K(s)(f)(z)| \leq 2 \sum_{k=0}^{\infty} |a_k|k^3r^k|t - s|.
\]

(iv) The families \( \{T_K(t)\}_{t \geq 0} \) and \( \{\tilde{T}_K(t)\}_{t \geq 0} \) are \((C_0)\)-semigroups of linear operators, locally equicontinuous (that is, equicontinuous for \( t \in [0, a] \), for some \( a > 0 \)) on \( A^*(D_R) \). For each \( f \in A^*(D_R) \), the corresponding Cauchy problems

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial \varphi^3} &= 0, & (t, z) &\in \mathbb{R}_+ \times D_R \setminus \{0\}, \\
\quad u(0, z) &= f(z), & z &\in \overline{D_R}
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial w}{\partial t} &= \frac{\partial^3 w}{\partial \varphi^3}, & (t, z) &\in \mathbb{R}_+ \times D_R \setminus \{0\}, \\
\quad u(0, z) &= f(z), & z &\in \overline{D_R}
\end{align*}
\]

are well-posed, with solutions given by

\[
\begin{align*}
u(t) &= T_K(t)(f) \in C^\infty(\mathbb{R}_+; A^*(D_R)), \\
w(t) &= \tilde{T}_K(t)(f) \in C^\infty(\mathbb{R}_+; A^*(D_R))
\end{align*}
\]

respectively.

Proof. We will prove the above statements only for the family \( \{T_K(t)\}_{t \geq 0} \) (the proof for \( \{\tilde{T}_K(t)\}_{t \geq 0} \) is the same).

(i) Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \), for all \( z \in D_R \). For fixed \( z \in D_R \), we get

\[
f(ze^{-iu}) = \sum_{k=0}^{\infty} a_k e^{-iku} z^k.
\]

Since \( |a_k e^{-iku}| = |a_k| \), for all \( u \in \mathbb{R} \), and since \( \sum_{k=0}^{\infty} a_k z^k \) is absolutely convergent, the series \( \sum_{k=0}^{\infty} a_k e^{-iku} z^k \) is also uniformly convergent with respect to \( u \in \mathbb{R} \). Therefore, the latter can be integrated term by term. Using (19), we deduce

\[
T_K(t)(f)(z) = \sum_{k=0}^{\infty} a_k \left\{ \int_{-\infty}^{\infty} \left[ \frac{1}{\pi} \int_0^{\infty} \cos(\rho t) - t \varphi^3 d\rho \right] e^{-iku} du \right\} z^k
\]
where

\[
d_k(t) = \int_{-\infty}^{\infty} \left[ \frac{1}{\pi} \int_0^{\infty} \cos(u\tau - t\tau^3) d\tau \right] \cos(\omega ku) du
- i \int_{-\infty}^{\infty} \left[ \frac{1}{\pi} \int_0^{\infty} \cos(u\tau - t\tau^3) d\tau \right] \sin(\omega ku) du
= \int_{-\infty}^{\infty} \left[ \frac{1}{\pi} \int_0^{\infty} \cos(u\tau - t\tau^3) d\tau \right] \cos(\omega ku) du
+ \int_{0}^{\infty} \left[ \frac{1}{\pi} \int_0^{\infty} \cos(u\tau - t\tau^3) d\tau \right] \cos(\omega ku) du
- i \int_{-\infty}^{\infty} \left[ \frac{1}{\pi} \int_0^{\infty} \cos(u\tau - t\tau^3) d\tau \right] \sin(\omega ku) du
+ \int_{0}^{\infty} \left[ \frac{1}{\pi} \int_0^{\infty} \cos(u\tau - t\tau^3) d\tau \right] \sin(\omega ku) du
= \frac{1}{\pi} \int_{0}^{\infty} \left[ \int_{0}^{\infty} \left( \cos(u\tau + t\tau^3) + \cos(u\tau - t\tau^3) \right) d\tau \right] \cos(\omega ku) du
+ i \left\{ \int_{0}^{\infty} \left[ \int_{0}^{\infty} \left( \cos(u\tau + t\tau^3) - \cos(u\tau - t\tau^3) \right) d\tau \right] \sin(\omega ku) du \right\}.
\]

On the other hand, it is well-known that the Fourier transform of the Airy’s function is given by (see, e.g., [5] p. 87, Table 4.2)

\[
\int_{-\infty}^{\infty} \left[ \frac{1}{\pi} \int_0^{\infty} \cos(u\tau + \tau^3/3) d\tau \right] e^{-i\omega \mu} du = e^{i\omega \mu^3/3} =: I_1 + I_2,
\]
where

\[
I_1 := \frac{1}{\pi} \int_{0}^{\infty} \left[ \int_{0}^{\infty} \left( \cos(\tau^3/3 + u\tau) + \cos(\tau^3/3 - u\tau) \right) d\tau \right] \cos(\omega \mu u) du,
I_2 := \frac{i}{\pi} \left\{ \int_{0}^{\infty} \left[ \int_{0}^{\infty} \left( \cos(\tau^3/3 + u\tau) - \cos(\tau^3/3 - u\tau) \right) d\tau \right] \sin(\omega \mu u) du \right\}.
\]

Now, by a change of variable \( \tau = (3t)^{1/3}\eta \), and then another \( u = \frac{\eta}{(3t)^{1/3}\tau} \), (\( t > 0 \) is a fixed parameter), simple calculations yield

\[
I_1 + I_2 = \frac{1}{\pi} \int_{0}^{\infty} \left[ \int_{0}^{\infty} \left( \cos(t\eta^3 + \eta \mu) + \cos(t\eta^3 - \eta \mu) \right) d\eta \right] \cos(\omega \mu v / (3t)^{1/3}) dv
+ \frac{i}{\pi} \int_{0}^{\infty} \left[ \int_{0}^{\infty} \left( \cos(t\eta^3 + \eta \mu) - \cos(t\eta^3 - \eta \mu) \right) d\eta \right] \sin(\omega \mu v / (3t)^{1/3}) dv
= e^{i\omega \mu^3 / 3}.
\]

Choosing \( \omega = (3t)^{1/3} \), and taking into account that \( I_1 + I_2 = d_k(t) \), we easily arrive at

\[
d_k(t) = e^{itk^3}.
\]
This proves the analyticity of $T_K (\cdot) (f) (z)$, as function of $z \in D_R$. Finally, the relation $T_K (0) = I$ is immediate in view of (21).

(ii) Let $|z| \leq r$. We obtain

$$|T_K (t) (f) (z) - f(z)| = \left| \sum_{k=0}^{\infty} a_k z^k [e^{ik^3 t} - 1] \right|$$

$$= \left| \sum_{k=0}^{\infty} a_k z^k [-2 \sin^2 (k^3 t/2) + i \sin (k^3 t/2) \cos (k^3 t/2)] \right|$$

$$\leq \sum_{k=0}^{\infty} |a_k| r^k |2 \sin (k^3 t/2)| + \sum_{k=0}^{\infty} |a_k| r^k |2 \sin (k^3 t/2)|$$

$$\leq \frac{t^2}{2} \sum_{k=0}^{\infty} |a_k| r^k + |t| \sum_{k=0}^{\infty} |a_k| r^k,$$

since $|\sin(x)| \leq |x|$, for all $x \in \mathbb{R}$.

(iii) We have

$$|T_K (t) (f) (z) - T_K (s) (f) (z)|$$

$$= \left| \sum_{k=0}^{\infty} a_k z^k [e^{ik^3 t} - e^{ik^3 s}] \right|$$

$$= \left| \sum_{k=0}^{\infty} [\cos (k^3 t) - \cos (k^3 s) + i(\sin (k^3 t) - \sin (k^3 s))] \right|$$

$$\leq 4 \sum_{k=0}^{\infty} |a_k| r^k \left| \sin \left( \frac{k^3 (t-s)}{2} \right) \right|$$

$$\leq 2 \sum_{k=0}^{\infty} |a_k| r^k |t-s|.$$

(iv) From (i), it is immediate that $T_K (t+s) = T_K (t) T_K (s)$, for all $t, s \in \mathbb{R}_+$. The strong continuity of $T_K (t)$ follows from (iii). We can argue as in the proof of Theorem 6.2.1 to deduce the first part of the statement in (iv). Let $f \in A^+ (D_R)$. We can compute the generators of the semigroups $T_K (t)$ and $\tilde{T}_K (t)$, $t \in \mathbb{R}_+$, respectively, as follows:

$$\left( \frac{d}{dt} T_K (t) (f) (z) \right)_{t=0} = i \sum_{k=0}^{\infty} k^3 a_k e^{ik^3 t} z^k = -\frac{\partial^3}{\partial \varphi^3} T_K (t) (f) (z)$$

and

$$\left( \frac{d}{dt} \tilde{T}_K (t) (f) (z) \right)_{t=0} = -i \sum_{k=0}^{\infty} k^3 a_k e^{-ik^3 t} z^k = \frac{\partial^3}{\partial \varphi^3} \frac{d}{dt} \tilde{T}_K (t) (f) (z),$$

for all $z = re^{i\varphi} \in D_R \setminus \{0\}$. The proof of the theorem is complete. \qed
In what follows, the linearized Korteweg-de Vries equation from (5) is complexified by replacing $x \in \mathbb{R}$ with $z \in \Omega \subset \mathbb{C}$ directly in the equations. More precisely, we aim to study the following initial value problem

$$
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^3 u}{\partial z^3}, \\
( t, z) &\in \mathbb{R}_+ \times \Omega, \\
\lim_{t \to 0} u(t, z) &= f(z), \\n&\quad z \in \Omega.
\end{align*}
$$

(24)

We look for classical solutions of (24) which belong to the class:

$$
u \in C^1(\mathbb{R}_+;A^\circ(\mathbb{S}_r)),$n$$

(25)

where

$$A^\circ(\mathbb{S}_r) := \{ f : \mathbb{S}_r \to \mathbb{C} : f \text{ is analytic in } \mathbb{S}_r \},$$

i.e., $f$ is analytic in a domain that contains the closure $\overline{\mathbb{S}_r}$ of $\mathbb{S}_r := \{ z = x + iy \in \mathbb{C} : x \in \mathbb{R}, |y| < r \}$.

**Theorem 3.2.** Let $f \in A^\circ(\mathbb{S}_r)$ such that the following are satisfied:

(i) $f'$ is bounded on $\mathbb{S}_r$;

(ii) for all $|y| \leq r$, the integral $F(s, y) = \int_{-\infty}^{+\infty} e^{-sx} f(x + iy) dx$ is absolutely convergent for $s = c_1 + i\tau$;

(iii) for all $|y| \leq r$, the integral $G(s, y) = \int_{-\infty}^{+\infty} e^{-sx} f'(x + iy) dx$ is absolutely convergent for $s = c_1 + i\tau$;

(iv) for all $|y| \leq r$, $\int_{-\infty}^{+\infty} |F(c_1 + i\tau, y)| d\tau < +\infty$.

Setting

$$T_{KV}(t)(f)(z) := \int_{-\infty}^{+\infty} K(u, t)f(z - u)du, \quad (t, z) \in \mathbb{R}_+ \times \mathbb{S}_r,$n$$

there holds $u(t) = T_{KV}(t)(f) \in C^\infty(\mathbb{R}_+;A^\circ(\mathbb{S}_r))$, and $T_{KV}(t)(f)$ solves the initial value problem (24).

**Proof.** Let $f \in A^\circ(\mathbb{S}_r)$ and decompose $f(z) = U(x, y) + iV(x, y)$, with $U$ and $V$ having continuous partial derivatives of first order. Moreover, $U, V$ satisfy the Cauchy-Riemann conditions at any $(x, y) \in \mathbb{S}_r$. We can write

$$T_{KV}(t)(f)(z) = \int_{-\infty}^{+\infty} K(u, t)U(x - u, y)du + i \int_{-\infty}^{+\infty} K(u, t)V(x - u, y)du \quad := T_1(t, x, y) + iT_2(t, x, y).$$

**Step 1.** Let $z = x + iy \in \mathbb{S}_r$. Since

$$f'(z) = \frac{\partial U}{\partial x}(x, y) + i\frac{\partial V}{\partial x}(x, y),$$

(26)
we note that
\[ f'''(z) = \frac{\partial^3 U}{\partial x^3}(x, y) + i \frac{\partial^3 V}{\partial x^3}(x, y). \]

We claim that for any fixed \( |y| \leq r \), conditions (i)-(ii) in the statement of Theorem 7.1.1, are fulfilled for \( T_1 \) and \( T_2 \), with respect to \( \{ t, x \} \in \mathbb{R}_+ \times \mathbb{R} \). As a consequence,

\[
\left\{ \begin{array}{l}
\frac{\partial T_1}{\partial t}(t, x, y) = \frac{\partial^2 T_1}{\partial x^2}(t, x, y), (t, x, y) \in \mathbb{R}_+ \times \mathbb{S}_r, \\
\lim_{t \to 0^+} T_1(t, x, y) = U(x, y), (x, y) \in \mathbb{S}_r.
\end{array} \right.
\]

\[
\left\{ \begin{array}{l}
\frac{\partial T_2}{\partial t}(t, x, y) = \frac{\partial^2 T_2}{\partial x^2}(t, x, y), (t, x, y) \in \mathbb{R}_+ \times \mathbb{S}_r, \\
\lim_{t \to 0^+} T_2(t, x, y) = V(x, y), (x, y) \in \mathbb{S}_r,
\end{array} \right.
\]

and, therefore, \( T_{KV}(t)(f) \) solves (24). Indeed, by the first hypothesis above (see (i)), there exists \( M > 0 \) such that
\[ \sup \{ |f'(z)| : z \in \mathbb{S}_r \} = \| f' \|_{\mathbb{S}_r} \leq M. \]

Now, from (20), \( \frac{\partial U}{\partial x}(x, y) \) and \( \frac{\partial V}{\partial x}(x, y) \) are bounded on \( \mathbb{S}_r \). Therefore, for any fixed \( |y| \leq r \), \( U(x, y) \) and \( V(x, y) \) are continuous and of bounded variation with respect to \( x \in \mathbb{R} \). Setting
\[ F_1(s, y) = \int_{-\infty}^{+\infty} e^{-sx} U(x, y) dx, \quad F_2(s, y) = \int_{-\infty}^{+\infty} e^{-sx} V(x, y) dx, \]

clearly, \( F(s, y) = F_1(s, y) + iF_2(s, y) \), and by virtue of (ii), both \( F_1(s, y) \) and \( F_2(s, y) \) are absolutely convergent, for a fixed (but otherwise arbitrary) \( |y| \leq r \). Furthermore, in view of (iv), we deduce
\[ \int_{-\infty}^{+\infty} |F_1(c_1 + i\tau, y)| d\tau < +\infty, \quad \int_{-\infty}^{+\infty} |F_2(c_1 + i\tau, y)| d\tau < +\infty. \]

Therefore, we can apply Theorem 1.1 to the functions \( T_1 \) and \( T_2 \), respectively. This yields the above claim.

**Step 2.** Clearly, \( T_{KV}(t)(f) \) belongs to the class (25) for any \( f \in \mathcal{A}(\mathbb{S}_r) \). Indeed, the fact that \( \partial_x T_1(\cdot, x, y) \) and \( \partial_x T_2(\cdot, x, y) \) are continuous on \( \mathbb{S}_r \), was already proved in Step 1. The existence and continuity of the partial derivatives \( \partial_y T_1(\cdot, x, y) \), \( \partial_y T_2(\cdot, x, y) \) follow from condition (iii). Finally, the functions \( T_i(\cdot, x, y), i = 1, 2 \), also satisfy the Cauchy-Riemann equations since \( U, V \) do. The proof is finished. \( \Box \)

**Remark 3.1.** A simple example of boundary data in (24) that satisfies all the hypotheses of Theorem 3.2 is
\[ f(z) = e^{-z^2}. \]

In this case, one can prove that \( F(s, y) = \sqrt{\pi} e^{s^2/4} \cos(sy). \)

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References


Existence and stability in the $\alpha$-norm for nonlinear neutral partial differential equations with finite delay

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ABSTRACT

In this work, we study the existence, regularity and stability of solutions for some nonlinear class of partial neutral functional differential equations. We assume that the linear part generates a compact analytic semigroup on a Banach space $X$, the delayed part is assumed to be continuous with respect to the fractional power of the generator. For illustration, some application is provided for some model with diffusion and nonlinearity in the gradient.

RESUMEN

En este trabajo estudiamos la existencia, regularidad y estabilidad de soluciones para una clase de ecuaciones diferenciales parciales funcionales neutrales. Asumimos que la parte lineal genera un semigrupo compacto analítico en un espacio de Banach $X$, la parte retardada se asume continua respecto de la potencia fraccional del generator. Como ejemplo, se muestra una aplicación para un modelo con difusión y no linealidad en el gradiente.

Keywords and Phrases: Neutal equation; Analytic semigroup; Fractional power; Mild solution; Strict solution.

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1 Introduction

In this paper, we study the existence, regularity and stability of solutions in the $\alpha$-norm for partial differential equations with finite delay. The following model provides an example of such a situation

\[
\begin{align*}
\frac{\partial}{\partial t} \left[ v(t, x) - qv(t - r, x) + g(\frac{\partial}{\partial x} v(t - r, x)) \right] &= \frac{\partial^2}{\partial x^2} \left[ v(t, x) - qv(t - r, x) + g(\frac{\partial}{\partial x} v(t - r, x)) \right] + f(v(t - r, x) - qv(t - r, x)) \\
&= A\phi(t, x) + f(v(t - r, x) - qv(t - r, x)) \\
&= : A_x(v(t, x)) + f(v(t - r, x) - qv(t - r, x)) \\
\end{align*}
\]

(1)

where \( q, r \) are positive constants, the initial data \( v_0 \) is given function and \( f, g \) are continuous functions. The previous system can be written as a neutral partial differential equation of the following form

\[
\begin{align*}
\frac{d}{dt}[x(t) - G(t, x_t)] &= -A[x(t) - G(t, x_t)] + F(t, x_t) \quad \text{for} \ t \geq 0, \\
x_0 = \phi \in C_\alpha,
\end{align*}
\]

(2)

where \(-A\) generates an analytic semigroup \((T(t))_{t \geq 0}\) on a Banach space \( X, C := \mathbb{C}([-r, 0], D(A^\alpha)), r > 0, \) and \( 0 < \alpha < 1 \), denotes the space of continuous functions from \([-r, 0]\) into \( D(A^\alpha) \), and the operator \( A^\alpha \) is the fractional \( \alpha \)-power of \( A \). This operator \((A^\alpha, D(A^\alpha))\) will be describe later. For \( x \in \mathbb{C}([-r, b], D(A^\alpha)), b > 0, \) and \( t \in [0, b] \), \( x_t \) denotes, as usual, the element of \( C_\alpha \) defined by \( x_t(\theta) = x(t + \theta) \) for \( \theta \in [-r, 0] \). \( G \) and \( F \) are continuous functions from \( \mathbb{R}_+ \times C_\alpha \) with values respectively in \( X_\alpha \) and \( X \).

This work was motivated by \[4, 18\]. In \[4\] the authors have developed a basic theory of partial neutral functional differential equations in fractional power spaces, they proved the existence and regularity of the solution of Eq. \( 2 \), but only in the case where \( G : C_\alpha \rightarrow D(A^\alpha) \) is a bounded linear operator. They considered the following neutral partial differential equation

\[
\begin{align*}
\frac{d}{dt}[D(x_t)] &= -AD(x_t) + F(x_t) \quad \text{for} \ t \geq 0, \\
x_0 = \phi \in C_\alpha,
\end{align*}
\]

(3)

where \( D \) is a bounded linear operator from \( C_\alpha \) into \( X_\alpha \) defined by \( D\phi = \phi(0) - D_0\phi \), for \( \phi \in C_\alpha \), where \( D_0 \) is a bounded linear operator given by:

\[
D_0\phi = \int_{-r}^{0} d\eta(\theta)\phi(\theta) \quad \text{for} \ \phi \in C_\alpha,
\]

and \( \eta : [-r, 0] \rightarrow \mathcal{L}(X_\alpha) \) is of bounded variation and non-atomic at zero. That is

\[
\text{var}_{[-\varepsilon, 0]}(\eta) \rightarrow 0 \quad \text{as} \ \varepsilon \rightarrow 0.
\]
Which $F$ is a globally Lipschitz continuous mapping from $C_\alpha$ into $D(A^\alpha)$, and if $x \in D(A^\alpha)$ and $\theta \in [-r, 0]$ then $\eta(\theta)x \in D(A^\alpha)$ and $A^\alpha \eta(\theta)x = \eta(\theta)A^\alpha x$.

It is well known, that if the phase space $C_\alpha$ is the space of all continuous functions from $[-r, 0]$ into $X$ (i.e. $\alpha = 0$), Equation (3) has been studied by several authors. For more details, we refer to the book of Wu [29]. For example, Wu and Xia considered in [30] a system of partial neutral functional differential-difference equations defined on the unit circle $S^1$, which is a model for a continuous circular array of resistively coupled transmission lines with mixed initial boundary conditions. They obtained equations of the form

$$\frac{\partial}{\partial t}[x(\cdot, t) - qx(\cdot, t - r)] = K \frac{\partial^2}{\partial \xi^2}[x(\cdot, t) - qx(\cdot, t - r)] + f(x_t) \quad \text{for } t \geq 0,$$

where $\xi, \theta \in S^1$, $K$ a positive constant and $0 \leq q < 1$. The space of initial data was chosen to be $C([-r, 0]; H^1(S^1))$. Motivated by this work, Hale presented, in [19, 20], the basic theory of existence and uniqueness, and properties of the solution operator, as well as Hopf bifurcation and conditions for the stability and instability of periodic orbits for a more general class of PNFDE on the unit circle $S^1$. For the sake of comparison, let us briefly restate the equations considered by Hale in [19, 20]. If $\phi \in C([-r, 0]; H^1(S^1))$, we write it as $\phi(\xi, \theta)$ for $\xi \in S^1$ and $\theta \in [-r, 0]$. For any function $f \in C^{k+1}([-r, 0]; H^1(S^1); L^2(S^1))$ we define by $f(\phi)(\xi) = f(\phi(\xi, .), \xi) \in S^1$. Let $D \in L(C([-r, 0]; H^1(S^1); L^2(S^1))$ be defined by

$$\begin{cases}
\tilde{D} \psi = \psi(0) - \tilde{g}(\psi), \\
\tilde{g}(\psi) = \int_{-r}^{0} d\eta(\theta)\psi(\theta),
\end{cases}$$

where $\eta$ is of bounded variation and non-atomic at $0$.

We define $D \in L(C([-r, 0]; H^1(S^1); H^1(S^1))$ as

$$D(\phi)(\xi) = \tilde{D}(\phi(\xi, .)) \quad \text{for } \xi \in S^1.$$ 

Hale considered, in [19, 20], PNFDE of the form

$$\frac{\partial}{\partial t}Dx_t = K \frac{\partial^2}{\partial \xi^2}Dx_t + f(x_t) \quad \text{for } t \geq 0,$$ (4)

with $C([-r, 0]; H^1(S^1))$ as the space of initial data. He considered the Laplace operator $A_0 = K \frac{\partial^2}{\partial \xi^2}$ with domain $H^2(S^1)$, which yields an operator generating an analytic semigroup. In [1, 2, 3], authors considered a natural generalization of the work of Hale [19, 20]. We extended the study to the case when the linear part of PNFDE is non-densely defined Hille-Yosida operator.

In [27], Travis and Webb investigated the local existence of mild solutions and strong solutions of Eq. (2) with respect to the $\alpha$-norm, but in the particular case when $G(\cdot, .) = 0$. The existence of strong solutions is considered when $F$ is locally Hölder continuous in both of its variables, also in [26], they studied the existence and regularity of mild solution when $F$ is Lipschitz continuous with
the \(X\)-norm.

Here, we assume that \(G\) is a nonlinear function and is defined in a smaller space than \(C_X\), that is \(C_{\alpha}\) for some \(0 < \alpha < 1\), the space of continuous function from \([-r,0]\) into \(X_{\alpha}\), which will be specified later. We prove the existence of the mild and strict solution.

This paper is organized as follows. In Section 2, we recall some preliminary results about analytic semigroups and fractional power associated to its generator and the definition of the measure of noncompactness. After that, we start to prove the existence and uniqueness of mild solutions. In Section 3, we study the regularity of solution, we give sufficient conditions to get the existence of the strict solutions. In Section 4, we state some properties of the solution operator associated to the autonomous case of Eq. (2). Also, we investigate the stability near an equilibrium. Mainly, we prove that the equilibrium of the solution semigroup associated to the autonomous case is locally exponentially stable when its linearized solution semigroup around this equilibrium is exponentially stable. Finally, to illustrate our results, we give in Section 5 an application to a reaction diffusion equation.

## 2 Existence of mild solutions

Let \((X, \| \cdot \|)\) be a Banach space, and \(\alpha\) be a constant such that \(0 < \alpha < 1\) and \(-A\) be the infinitesimal generator of a bounded analytic semigroup of linear operator \(\{T(t)\}_{t \geq 0}\) on \(X\). We assume without loss of generality that \(0 \in \rho(A)\). Note that if the assumption \(0 \in \rho(A)\) is not satisfied, one can substitute the operator \(A\) by the operator \((A - \sigma I)\) with \(\sigma\) large enough such that \(0 \in \rho(A - \sigma)\).

This allows us to define the fractional power \(A^{\alpha}\) for \(0 < \alpha < 1\), as a closed linear invertible operator with domain \(D(A^{\alpha})\) dense in \(X\). The closeness of \(A^{\alpha}\) implies that \(D(A^{\alpha})\), endowed with the graph norm of \(A^{\alpha}\), \(\|x\| = \|x\| + \|A^{\alpha}x\|\), is a Banach space. Since \(A^{\alpha}\) is invertible, its graph norm \(\|\cdot\|\) is equivalent to the norm \(\|x\|_{\alpha} = \|A^{\alpha}x\|\). Thus, \(D(A^{\alpha})\) equipped with the norm \(\|\cdot\|_{\alpha}\), is a Banach space, which we denote by \(X_{\alpha}\). The space \(C_{\alpha} := C([-r,0],X_{\alpha})\), \(r > 0\) denotes the space of continuous functions from \([-r,0]\) into \(X_{\alpha}\) endowed with the uniform norm topology:

\[
\|\varphi\|_{\alpha} := \sup_{0 \in [-r,0]} \|\varphi(t)\|_{\alpha} \quad \text{for} \quad \varphi \in C_{\alpha}.
\]

Also, the following properties are well known.

**Theorem 2.1.** ([24]) Let \(0 < \alpha < 1\). Assume that the operator \(-A\) is the infinitesimal generator of an analytic semigroup \(\{T(t)\}_{t \geq 0}\) on the Banach space \(X\) satisfying \(0 \in \rho(A)\). Then we have

\begin{enumerate}
\item[i)] \(T(t) : X \rightarrow D(A^{\alpha})\) for every \(t > 0\),
\item[ii)] \(T(t)A^{\alpha}x = A^{\alpha}T(t)x\) for every \(x \in D(A^{\alpha})\) and \(t \geq 0\),
\item[iii)] for every \(t > 0\), \(A^{\alpha}T(t)\) is bounded on \(X\) and there exist \(M_{\alpha} > 0\) and \(\delta > 0\) such that
\[
\|A^{\alpha}T(t)\| \leq M_{\alpha}e^{-\delta t}t^{-\alpha} \leq M_{\alpha}t^{-\alpha} \quad \text{for} \quad t > 0,
\]
\end{enumerate}
iv) If $0 < \alpha \leq \beta < 1$, then $D(A^\beta) \hookrightarrow D(A^\alpha)$.

v) There exists $N_\alpha > 0$ such that
$$\| (T(t) - I)A^{-\alpha} \| \leq N_\alpha t^\alpha \quad \text{for } t > 0.$$  

vi) If $T(t)$ is compact for each $t > 0$, then $A^{-\alpha}$ is compact.

Now, we propose to find the existence of a mild solution for problem (2) using the Sadovskii’s fixed point theorem. Then, we obtain the uniqueness result of the solution by adding a hypothesis of Lipschitz continuous on $F$.

Let $E$ be a Banach space. We introduce the Kuratowski measure of noncompactness $\chi(\Omega)$ of a set $\Omega \subset E$ by
$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \ \text{has a finite cover of diameter} \ < \varepsilon\}.$$

**Lemma 2.1.** \cite{10} Let $E$ be a Banach space and $B, C \subseteq E$ be bounded set. Then, the following properties are true:

1. $B$ is relatively compact if and only if $\chi(B) = 0$.
2. $\chi(B + C) \leq \chi(B) + \chi(C)$, where $B + C = \{x + y : x \in B, \ y \in C\}$.
3. Every Lipschitz continuous function $K$ from $C$ to $F$ satisfies:
$$\chi[K(\Omega)] \leq \text{lip}_K \chi(\Omega),$$

where $\text{lip}_K$ decides the smallest Lipschitz constant of $K$.

**Definition 2.2.** \cite{25} A mapping $K$ from a set $C$ in a Banach space $E$ is called a condensing operator if it is continuous and for every bounded noncompact set $\Omega \subseteq C$ the inequality holds
$$\chi[K(\Omega)] < \chi(\Omega).$$

**Theorem 2.2.** \cite{25} (Sadovskii’s fixed point theorem). If a condensing mapping $K$ maps a bounded convex closed set $C$ in a Banach space $E$ into itself, then $K$ has at least one fixed point in $T$.

First of all, we study the existence of mild solutions, in order to do that, we assume the following assumptions.

(H0) The operator $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t \geq 0}$ on the Banach space $X$, moreover, we assume that $0 \in \rho(A)$.

(H1) The semigroup $(T(t))_{t \geq 0}$ is compact on $X$ for $t > 0$. It means that $T(t)$ is compact on $X$ for $t > 0$. 

**H2** \( G : [0, a] \times C_\alpha \rightarrow X_\alpha \) is continuous and for each \( \alpha > 0 \) there exists \( 0 < L_g < 1 \) such that \( |G(t, \varphi) - G(t, \psi)|_\alpha \leq L_g \|\varphi - \psi\|_\alpha \) for every \( t \in [0, a] \) and \( \varphi, \psi \in C_\alpha \).

**H3** The function \( F : [0, a] \times C_\alpha \rightarrow X \) satisfies the following conditions

1. \( F : [0, a] \times C_\alpha \rightarrow X \) is continuous.
2. There exists a continuous nondecreasing function \( \beta : [0, a] \rightarrow \mathbb{R}_+ \) such that
   \[ \|F(t, \varphi)\| \leq \beta(t) \|\varphi\|_\alpha \] for \( (t, \varphi) \in [0, a] \times C_\alpha \).

**Definition 2.3.** A continuous function \( x : [-r, a] \rightarrow X_\alpha \), for \( a > 0 \) is said to be a mild solution of Eq. (2), if

1. \( x(t) = T(t)[\varphi(0) - G(0, \varphi)] + G(t, x_t) + \int_0^t T(t - s)F(s, x_s)\, ds \) for \( t \in [0, a] \),
2. \( x_0 = \varphi \).

**Definition 2.4.** A continuous function \( x : [-r, a] \rightarrow X_\alpha \) is said to be a strict solution of Eq. (2), if

1. \( x(.) - G(., x(.) ) \in C(\mathbb{R}_+, [0, a], X_\alpha) \),
2. \( \frac{d}{dt}(x(t) - G(t, x_t)) = -A(x(t) - G(t, x_t)) + F(t, x_t) \) for \( t \in [0, a] \),
3. \( x_0 = \varphi \).

Now, we state our first result.

**Theorem 2.3.** Assume that the hypothesis \((H0)-(H3)\) hold. Let \( \varphi \in C_\alpha \). Assume that the following condition holds

\[ L_g + M_\alpha \int_0^a \frac{\beta(s)}{(a - s)_\alpha} \, ds < 1. \]  \( (5) \)

Then Eq. (2) has at least one mild solution on \( [0, a] \).

**Proof.** Let \( k > \|\varphi\|_\alpha \). We define the following set

\[ B_k = \{ x \in C([0, a], X_\alpha) : x(0) = \varphi(0) \text{ and } |x|_\infty \leq k \}, \]

where \( |x|_\infty = \sup_{t \in [0, a]} |x(t)|_\alpha \). For \( x \in B_k \), define the mapping \( \tilde{x} : [-r, a] \rightarrow X_\alpha \) by

\[ \tilde{x}(t) = \begin{cases} x(t) & \text{for } t \in [0, a] \\ \varphi(t) & \text{for } t \in [-r, 0]. \end{cases} \]
The function $t \mapsto \tilde{x}_t$ is continuous from $[0,a]$ to $C_\alpha$.

Now, define the operator $K$ on $B_k$ by

$$K(x)(t) = T(t)(\varphi(0) - G(0, \varphi)) + G(t, \tilde{x}_t) + \int_0^t T(t - s)F(s, \tilde{x}_s)ds \quad \text{for } t \in [0,a].$$

It is sufficient to show that $K$ has a fixed point in $B_k$. We first show that there is a positive number $k > \|\varphi\|_\alpha$ such that $K(B_k) \subseteq B_k$. If not, then for each $k > \|\varphi\|_\alpha$, there exist $x_k \in B_k$ and $t_k \in [0,a]$ such that $\|Kx_k\|_{(t_k)} > k$. It follows that

$$k < \|Kx_k\|_{(t_k)}$$

$$\leq |T(t_k)(\varphi(0) - G(0, \varphi))| + |G(t_k, x_{t_k})| + \int_0^{t_k} |T(t_k - s)F(s, \tilde{x}_s)|_\alpha ds.$$ 

Let $M = \sup(\|T(t)\| : \ t \in [0,a])$. Then

$$k < M|\varphi(0) - G(0, \varphi)|_\alpha + |G(t_k, \tilde{x}_{t_k}) - G(t_k, 0)|_\alpha + |G(t_k, 0)|_\alpha$$

$$+ \int_0^{t_k} \frac{M_\alpha}{(t_k - s)^\alpha} \beta(s)\|\tilde{x}_s\|_\alpha ds.$$ 

Moreover $\|\tilde{x}\|_\alpha \leq k$ for all $s \in [0,a]$ and $x \in B_k$. Then, we obtain

$$k < M|\varphi(0) - G(0, \varphi)|_\alpha + |G(t_k, \tilde{x}_{t_k}) - G(t_k, 0)|_\alpha + |G(t_k, 0)|_\alpha$$

$$+ \int_0^{t_k} \frac{kM_\alpha}{(t_k - s)^\alpha} \beta(s)ds.$$ 

We shall show that the function $g : t \mapsto \int_0^t \frac{\beta(s)}{(t - s)^\alpha} ds$ is nondecreasing on $[0,a]$. Let $t, t' \in [0,a]$ be such that $t < t'$. Then we have

$$g(t) = \int_0^t \frac{\beta(t - s)}{s^\alpha} ds \leq \int_0^{t'} \frac{\beta(t' - s)}{s^\alpha} ds \leq \int_0^{t'} \frac{\beta(t' - s)}{(a - s)^\alpha} ds = g(t').$$ 

Therefore

$$k < M|\varphi(0) - G(0, \varphi)|_\alpha + L_g\|\tilde{x}_{t_k}\|_\alpha + \sup_{0 \leq s \leq a} |G(s, 0)|_\alpha + \int_0^a \frac{kM_\alpha}{(a - s)^\alpha} \beta(s)ds.$$ 

Dividing both sides by $k$ and taking the lower limit as $k \to +\infty$, then we get that

$$L_g + M_\alpha \int_0^a \frac{\beta(s)}{(a - s)^\alpha} ds \geq 1,$$

which contradicts (5). Consequently, there exists $k \geq 0$ such $K(B_k) \subseteq B_k$.

To prove that $K$ has at least a fixed point on $B_k$, we decompose $K$ as follows $K := K_1 + K_2$, where

$$K_1(x)(t) = G(t, \tilde{x}_t) \quad \text{for } t \in [0,a],$$

and

$$K_2(x)(t) = T(t)(\varphi(0) - G(0, \varphi)) + \int_0^t T(t - s)F(s, \tilde{x}_s)ds \quad \text{for } t \in [0,a].$$
We claim that $K_1$ is a strict contraction and $K_2$ is compact.

To see this, observe that for $t \in [0, a]$ and $x, y \in B_k$, we have by assumption (H2).

$$|K_1x(t) - K_1y(t)|_\alpha = |G(t, \tilde{x}_t) - G(t, \tilde{y}_t)|_\alpha$$

$$\leq L_g||\tilde{x}_t - \tilde{y}_t||_\alpha$$

$$\leq L_g|x - y|_\infty$$

Then $K_1$ is a strict contraction. We will prove now the continuity of $K_2$. Let $(x^n)_n \subset B_k$ with $x^n \to x$ in $B_k$. Then, the set $\Lambda = \{(s, \tilde{x}^n_s), (s, \tilde{x}_s) : s \in [0, a], n \geq 1\}$ is compact in $[0, a] \times C_\alpha$. By Heine's theorem implies that $F$ is uniformly continuous in $\Lambda$ and

$$|K_2(x^n) - K_2(x)|_\infty = \sup_{t \in [0, a]} \int_0^t A^\alpha T(t - s) \left( F(s, \tilde{x}^n_s) - F(s, \tilde{x}_s) \right) ds$$

$$\leq M_\alpha \int_0^a ds \sup_{s \in [0, a]} ||F(s, \tilde{x}^n_s) - F(s, \tilde{x}_s)|| \to 0 \text{ as } n \to +\infty.$$ 

and this yield the continuity of $K_2$, then the continuity of $K$ on $B_k$.

We next show that the operator $K_2$ is compact.

In order to apply Ascoli theorem we have to show that the set $\{K_2(x)(t) : x \in B_k\}$ is relatively compact for each $t \in ]0, a]$.

Let $t \in ]0, a]$ be fixed, and $\gamma > 0$ be such that $\alpha < \gamma < 1$. Then

$$\| (A^\gamma K_2(x))(t) \| \leq \| A^\gamma T(t)(\varphi(0) - G(0, \varphi)) \| + \| \int_0^t A^\gamma T(t - s)F(s, \tilde{x}_s)ds \|$$

$$\leq M_\gamma t^{-\gamma} \| \varphi(0) - G(0, \varphi) \| + kM_\gamma \int_0^t (t - s)^{-\gamma} \beta(s)ds \to +\infty.$$ 

Then for fixed $t \in ]0, a]$, \{$(A^\gamma K_2x)(t)$\} is bounded in $X$. Appealing (H1) and (vi) of Theorem 2.1 we deduce that $A^{-\gamma} : X \to X_\alpha$ is compact, it follows that $\{K_2(x)(t) : x \in B_k\}$ is relatively compact set in $X_\alpha$.

Next, we will show that $\{K_2x : x \in B_k\}$ is an equicontinuous family of functions. For $0 \leq t_1 < t_2 \leq a$, 

$$K_2x(t_2) - K_2x(t_1) = (T(t_2) - T(t_1))(\varphi(0) - G(0, \varphi)) + \int_{t_1}^{t_2} T(t_2 - s)F(s, \tilde{x}_s)ds$$

$$+ \int_0^{t_1} (T(t_2 - s) - T(t_1 - s))F(s, \tilde{x}_s)ds$$

$$= (T(t_2) - T(t_1))(\varphi(0) - G(0, \varphi)) + \int_{t_1}^{t_2} T(t_2 - s)F(s, \tilde{x}_s)ds$$

$$+ (T(t_2 - t_1) - 1) \int_0^{t_1} T(t_1 - s)F(s, \tilde{x}_s)ds.$$
We obtain that
\[
\|K_2x(t_2) - K_2x(t_1)\|_\alpha \leq \|(T(t_2) - T(t_1))A^\alpha(\varphi(0) - G(0, \varphi))\| + kM_\alpha\|\beta\|_\infty \int_{t_1}^{t_2} (t_2 - s)^{-\alpha} ds
+\|\int_{t_1}^{t_2} A^\alpha T(t_1 - s)F(s, \bar{x}_s)ds\|
\]

It’s clair to prove the first part tend to zero as \(|t_2 - t_1|\to 0\). Since for \(t_1 > 0\) the set \(\{\int_{0}^{t_1} A^\alpha T(t_1 - s)F(s, \bar{x}_s)ds : x \in B_k\}\)

is relatively compact in \(X\), there is a compact set \(\tilde{K}\) in \(X\) such that \(\int_{0}^{t_1} A^\alpha T(t_1 - s)F(s, \bar{x}_s)ds \in \tilde{K}\) for \(x \in B_k\).

By Banach-Steinhaus’s theorem, we have
\[
\|(T(t_2 - t_1) - I)\int_{0}^{t_1} A^\alpha T(t_1 - s)F(s, \bar{x}_s)ds\| \to 0 \text{ as } t_2 \to t_1,
\]
uniformly in \(x \in B_k\). Using similar argument for \(0 \leq t_2 < t_1 \leq a\), we can conclude that \(\{K_2x(t), x \in B_k\}\) is an equicontinuous. Using Ascoli-Arzla theorem, we deduce that \(K_2 : B_k \to B_k\) is compact, and \(K = K_1 + K_2\) is a condensing operator. By the Sadovskii’s fixed-point theorem \(2.2\), we conclude that \(K\) has at least one fixed point in \(B_k\), which is a mild solutions of Eq. \(2\) on \([0, a]\).

To prove result on uniqueness, we to assume that

\((H4)\) \(F : [0, a] \times C_\alpha \to X\) is continuous and Lipschitzian with respect to the second variable. Let \(L_f > 0\) be such that
\[
\|F(t, \psi_1) - F(t, \psi_2)\| \leq L_f\|\psi_1 - \psi_2\|_\alpha
\]
for every \(\psi_1, \psi_2 \in C_\alpha\) and \(t \in [0, a]\).

**Theorem 2.4.** Let \(\varphi \in C_\alpha\). If the assumptions \((H0)\), \((H2)\) and \((H4)\) are satisfied, then Eq. \(2\) has a unique mild solution provided that
\[
L_g + M_\alpha L_f \frac{\alpha}{\Gamma(1 - \alpha)} < 1.
\]

**Proof.** Consider the nonempty closed subset of \(C([0, a], X_\alpha)\) defined by
\[
\Omega(\varphi) := \{x \in C([0, a], X_\alpha) : x(0) = \varphi(0)\}.
\]
For \(x \in \Omega(\varphi)\), define the mapping \(\bar{x} : [-r, a] \to X_\alpha\) by
\[
\bar{x}(t) = \begin{cases} 
    x(t) & \text{for } t \in [0, a] \\
    \varphi(t) & \text{for } t \in [-r, 0].
\end{cases}
\]
Define the operator $K : \Omega(\varphi) \to \Omega(\varphi)$ by

$$K(x)(t) = T(t)(\varphi(0) - G(0, \varphi)) + G(t, \tilde{x}_t) + \int_0^t T(t - s)F(s, \tilde{x}_s)\,ds \text{ for } t \in [0, a].$$

We shall show that it is a strict contraction. Let $x, y \in \Omega(\varphi)$ and $t \in [0, a]$. Then

$$|Kx(t) - Ky(t)|_\alpha \leq |G(t, \tilde{x}_t) - G(t, \tilde{y}_t)|_\alpha + \int_0^t |T(t - s)[F(s, \tilde{x}_s) - F(s, \tilde{y}_s)]|_\alpha\,ds$$

$$\leq L_g \|\tilde{x}_t - \tilde{y}_t\|_\alpha + M_\alpha \int_0^t \|F(s, \tilde{x}_s) - F(s, \tilde{y}_s)\|(t - s)^{-\alpha}\,ds$$

$$\leq \left(L_g + M_\alpha L_f \frac{a^{1-\alpha}}{1-\alpha}\right)|x - y|_\alpha.$$

Then

$$|Kx - Ky|_\alpha \leq \left(L_g + M_\alpha L_f \frac{a^{1-\alpha}}{1-\alpha}\right)|x - y|_\alpha.$$

It follows that $K$ is a strict contraction since $L_g + M_\alpha L_f \frac{a^{1-\alpha}}{1-\alpha} < 1$. By the contraction principle, we conclude that there exists a unique fixed point $x$ for $K$ in $\Omega(\varphi)$, therefore Eq. (2) has a unique mild solution on $[-r, a]$. The proof is completed. \hfill \Box

3 \quad \textbf{Existence of strict solutions}

For the regularity of the integral solutions, we suppose moreover the following assumptions:

\textbf{(H5)} $G$ and $F$ are continuously differentiable and their partial derivatives are locally Lipschitzian with respect to the second argument in the sense that; for any compact set $K \subset [0, a] \times C_\alpha$,

there exist positive constants $L_1, L_2, L_3$ and $L_4$ such that

$$|D_1 G(t, \psi_1) - D_1 G(t, \psi_2)|_\alpha \leq L_1 \|\psi_1 - \psi_2\|_\alpha,$$

$$|D_2 G(t, \psi_1) - D_2 G(t, \psi_2)|_{L(C_\alpha, X_\alpha)} \leq L_2 \|\psi_1 - \psi_2\|_\alpha,$$

$$|D_1 F(t, \psi_1) - D_1 F(t, \psi_2)| \leq L_3 \|\psi_1 - \psi_2\|_\alpha,$$

$$|D_2 F(t, \psi_1) - D_2 F(t, \psi_2)|_{L(C_\alpha, X_\alpha)} \leq L_4 \|\psi_1 - \psi_2\|_\alpha,$$

for $(t, \psi_1), (t, \psi_2) \in K$ and $t \in [0, a]$. Where $D_1$ and $D_2$ are the partial derivatives with respect to the first and second argument.

**Theorem 3.1.** Assume that (H0), (H2), (H4), (H5) hold and condition \textbf{(7)} is true. Let $\varphi \in C^1([-r, 0], X_\alpha)$ be such that $\varphi(0) - G(0, \varphi) \in D(A)$ and

$$\varphi'(0) - D_1 G(0, \varphi) - D_2 G(0, \varphi)\varphi' = -A(\varphi(0) - G(0, \varphi)) + F(0, \varphi)$$

Then Eq. (3) has a unique strict solution on $[0, a]$. \hfill \Box
Moreover, we can see that

\[ \Lambda \]

\[ \text{For } P \]

Proof. Let \( x \) be the mild solution of Eq. (2). Consider the equation

\[
\begin{align*}
y(t) &= T(t)[-A(\varphi(0) - G(0, \varphi)) + F(0, \varphi)] + D_1 G(t, x_t) + D_2 G(t, x_t) y_t \\
&\quad + \int_0^t T(t-s)[D_1 F(s, x_s) + D_2 F(s, x_s) y_s] ds \\
y_0 &= \varphi' \in C_\alpha,
\end{align*}
\]

We claim that Eq. (8) has a unique solution on \([0, a]\). In fact, consider the operator \( P \) defined on \( \Lambda := \{ x \in C([-r, a]; X_\alpha) : x(t) = \varphi'(t) \text{ for } t \in [-r, 0] \} \) by

\[
P y(t) = \begin{cases} 
T(t)[-A(\varphi(0) - G(0, \varphi)) + F(0, \varphi)] + D_1 G(t, x_t) \\
\quad + D_2 G(t, x_t) y_t + \int_0^t T(t-s)[D_1 F(s, x_s) + D_2 F(s, x_s) y_s] ds & \text{for } t \in [0, a], \\
\varphi'(t) & \text{for } t \in [-r, 0].
\end{cases}
\]

Let \( u, v \in \Lambda \). Then for each \( t \in [0, a] \), we have

\[
|Pu(t) - Pv(t)|_\alpha \leq \|D_2 G(t, x_t)\|_{\mathcal{L}(C_\alpha, X_\alpha)} \|u_t - v_t\|_\alpha \\
\quad + M_\alpha \int_0^t \|D_2 F(s, x_s)\|_{\mathcal{L}(C_\alpha, X_\alpha)} \|u_s - v_s\|_\alpha \frac{ds}{(t-s)^{\alpha}} \\
\quad \leq \left( L + M_\alpha L \frac{a^{1-\alpha}}{\Gamma(1-\alpha)} \right) \|u - v\|_{L_\infty}.
\]

Then \( P \) is a strict contraction. Consequently, it has a unique mild solution \( y \).

Define \( z : [-r, a] \rightarrow X_\alpha \) by

\[
z(t) = \begin{cases} 
\varphi(0) + \int_0^t y(s) ds & \text{for } t \in [0, a], \\
\varphi'(t) & \text{for } t \in [-r, 0],
\end{cases}
\]

we will show that \( z(t) = x(t) \) on \([0, a]\).

For \( t \in [0, a] \), we have

\[
z(t) = \varphi(0) + \int_0^t T(s)(-A)(\varphi(0) - G(0, \varphi)) ds + \int_0^t T(s)F(0, \varphi) ds \\
\quad + \int_0^t D_1 G(s, x_s) + D_2 G(s, x_s) y_s ds \\
\quad + \int_0^t \int_0^s T(s-\tau)D_1 F(\tau, x_{\tau}) + D_2 F(\tau, x_{\tau}) y_{\tau} d\tau ds.
\]

Moreover, we can see that

\[
z_t = \varphi + \int_0^t y_s ds \text{ for } t \in [0, a].
\]

(9)
Then \( t \mapsto z_t \) and \( t \mapsto \int_0^t T(t - s)F(s, z_s)\,ds \) are continuously differentiable on \([0, a]\) and satisfy

\[
\frac{d}{dt} \int_0^t T(t - s)F(s, z_s)\,ds = T(t)F(0, \varphi) + \int_0^t T(t - s)[D_1F(s, z_s) + D_2F(s, z_s)y_s]\,ds,
\]

then (10) yields

\[
\int_0^t T(s)F(0, \varphi)\,ds = \int_0^t T(t - s)F(s, z_s)\,ds
\]

\[
- \int_0^s \int_0^t T(s - \tau)[D_1F(\tau, z_\tau) + D_2F(\tau, z_\tau)y_\tau]\,d\tau\,ds.
\]

On the other hand

\[
G(t, z_t) = G(0, \varphi) + \int_0^t \frac{d}{ds}G(s, z_s)\,ds
\]

\[
= G(0, \varphi) + \int_0^t D_1G(s, z_s) + D_2G(s, z_s)y_s\,ds.
\]

Then

\[
z(t) = T(t)(\varphi(0) - G(0, \varphi)) + G(t, z_t) - \int_0^t D_1G(s, z_s) + D_2G(s, z_s)y_s\,ds
\]

\[
+ \int_0^t T(t - s)F(s, z_s)\,ds - \int_0^t \int_0^s T(s - \tau)[D_1F(\tau, z_\tau) + D_2F(\tau, z_\tau)y_\tau]\,d\tau\,ds
\]

\[
+ \int_0^t (D_1G(s, x_s) + D_2G(s, x_s)y_s)\,ds + \int_0^t \int_0^s T(s - \tau)[D_1F(\tau, x_\tau) + D_2F(\tau, x_\tau)y_\tau]\,d\tau\,ds.
\]

Therefore

\[
|z(t) - x(t)|_\alpha \leq |G(t, z_t) - G(t, x_t)|_\alpha + \int_0^t |D_1G(s, z_s) - D_1G(s, x_s)|_\alpha\,ds
\]

\[
+ \int_0^t |D_2G(s, z_s)y_s - D_2G(s, x_s)y_s|_\alpha\,ds
\]

\[
+ \int_0^t |T(t - s)[F(s, z_s) - F(s, x_s)]|_\alpha\,ds
\]

\[
+ \int_0^t \int_0^s |T(s - \tau)[D_1F(\tau, z_\tau) - D_1F(\tau, x_\tau)]|_\alpha\,d\tau\,ds
\]

\[
+ \int_0^t \int_0^s |T(s - \tau)[D_2F(\tau, z_\tau)y_\tau - D_2F(\tau, x_\tau)y_\tau]|_\alpha\,d\tau\,ds.
\]

Note that the sets \( \{(s, z_s) : s \in [0, a]\} \) and \( \{(s, x_s) : s \in [0, a]\} \) are compacts in \([0, a] \times C_\alpha\), since the mapping \( t \mapsto z_t \) and \( t \mapsto x_t \) are continuous on \([0, a]\). Then, we deduce that

\[
\|D_1G(s, z_s) - D_1G(s, x_s)\|_{C(\alpha, X_\alpha)} \leq L_1\|z_s - x_s\|_\alpha,
\]

\[
\|D_2G(s, z_s) - D_2G(s, x_s)\|_{C(\alpha, X_\alpha)} \leq L_2\|z_s - x_s\|_\alpha,
\]

\[
\|D_1F(s, z_s) - D_1F(s, x_s)\| \leq L_3\|z_s - x_s\|_\alpha,
\]

\[
\|D_2F(s, z_s) - D_2F(s, x_s)\|_{C(\alpha, X)} \leq L_4\|z_s - x_s\|_\alpha.
\]
for all \( s \in [0, \alpha] \), \( x \in \Lambda \) and \( z \) given in (3). Let \( L = \max \{ L_{f}, L_{1}, L_{2}, L_{3}, L_{4} \} \). Then

\[
|z(t) - x(t)|_{\alpha} \leq L_{g} + L \left( t + \|y\|_{\infty} t + \frac{M_{\alpha}}{1 - \alpha} t^{1 - \alpha} + \frac{M_{\alpha}}{(1 - \alpha)(2 - \alpha)} t^{2 - \alpha} \right) \sup_{0 \leq s \leq t} |z(s) - x(s)|_{\alpha}.
\]

We can choose \( t_{0} \in [0, \alpha] \) such that

\[
L_{g} + L \left( t_{0} + \|y\|_{\infty} t_{0} + \frac{M_{\alpha}}{1 - \alpha} t_{0}^{1 - \alpha} + \frac{M_{\alpha}}{(1 - \alpha)(2 - \alpha)} t_{0}^{2 - \alpha} + \frac{M_{\alpha}}{\alpha} \|y\|_{\infty}^{\alpha} t_{0}^{2 - \alpha} \right) < 1.
\]

we deduce that \( x = z \) on \([0, t_{0}]\). We claim that \( x(t) = z(t) \) for \( t \in [0, \alpha] \). We proceed by contradiction and assume that there exists \( t_{1} \in [0, \alpha] \) such that \( x(t_{1}) \neq z(t_{1}) \). Let \( t^{*} \) be the smallest number such that \( x(t) \neq z(t) \). Then

\[
t^{*} = \inf \{ t \in [0, \alpha] : |z(t) - x(t)|_{\alpha} > 0 \}.
\]

By continuity, one has \( x(t) = z(t) \) for \( t \in [0, t^{*}] \) and there exists \( \varepsilon > 0 \) such that

\[
|z(t) - x(t)|_{\alpha} > 0 \text{ for } t \in [t^{*}, t^{*} + \varepsilon].
\]

It follows for \( t \in [t^{*}, t^{*} + \varepsilon] \) that

\[
|z(t) - x(t)|_{\alpha} \leq |G(t, z_{t}) - G(t, x_{t})|_{\alpha} + \int_{t^{*}}^{t} |D_{1} G(s, z_{s}) - D_{1} G(s, x_{s})|_{\alpha} ds
\]

\[
+ \int_{t^{*}}^{t} |D_{2} G(s, z_{s}) y_{s} - D_{2} G(s, x_{s}) y_{s}|_{\alpha} ds
\]

\[
+ \int_{t^{*}}^{t} |T(t - s)(F(s, z_{s}) - F(s, x_{s}))|_{\alpha} ds
\]

\[
+ \int_{t^{*}}^{t} \int_{t^{*}}^{s} |T(s - \tau)(D_{1} F(\tau, z_{\tau}) - D_{1} F(\tau, x_{\tau}))|_{\alpha} d\tau ds
\]

\[
+ \int_{t^{*}}^{t} \int_{t^{*}}^{s} |T(s - \tau)(D_{2} F(\tau, z_{\tau}) y_{\tau} - D_{2} F(\tau, x_{\tau}) y_{\tau})|_{\alpha} d\tau ds.
\]

Consequently,

\[
|z(t) - x(t)|_{\alpha} \leq L_{g} + L \left( \varepsilon + \|y\|_{\infty} \varepsilon + \frac{M_{\alpha}}{1 - \alpha} \varepsilon^{1 - \alpha} + \frac{M_{\alpha}}{(1 - \alpha)(2 - \alpha)} \varepsilon^{2 - \alpha} \right) \sup_{t^{*} \leq s \leq t^{*} + \varepsilon} |z(s) - x(s)|_{\alpha}.
\]

If we choose \( \varepsilon \) such that

\[
L_{g} + L \left( \varepsilon + \|y\|_{\infty} \varepsilon + \frac{M_{\alpha}}{1 - \alpha} \varepsilon^{1 - \alpha} + \frac{M_{\alpha}}{(1 - \alpha)(2 - \alpha)} \varepsilon^{2 - \alpha} \right) < 1
\]

then \( x(t) = z(t) \) for \( t \in [t^{*}, t^{*} + \varepsilon] \) which gives a contradiction. Consequently \( x(t) = z(t) \) for \( t \in [0, \alpha] \) and \( t \mapsto x_{t} \) is continuously differentiable in \([0, \alpha]\) and \( t \mapsto F(t, x_{t}) \in C^{1}([0, \alpha], X) \). To end the proof, we use the following lemma.
Lemma 3.1. [24] Let \( h : [0, a] \to X \) be continuously differentiable and \( u \) satisfy
\[
    u(t) = T(t)u_0 + \int_0^t T(t-s)h(s)ds \quad \text{for } t \in [0, a].
\]
If \( u_0 \in D(A) \), then \( u \) is continuously differentiable on \([0, a]\) and
\[
    u'(t) = -Au(t) + h(t) \quad \text{for } t \in [0, a].
\]

In our case, we have \( \varphi(0) - G(0, \varphi) \in D(A), \ t \mapsto F(t, x_t) \) is continuously differentiable on \([0, a]\) and
\[
    x(t) - G(t, x_t) = T(t)[\varphi(0) - G(0, \varphi)] + \int_0^t T(t-s)F(s, x_s)ds \quad \text{for } t \in [0, a].
\]
By Lemma 3.1, the mapping \( t \mapsto x(t) - G(t, x_t) \) is continuously differentiable on \([0, a]\) and for \( t \in [0, a]\),
\[
    \frac{d}{dt} [x(t) - G(t, x_t)] = -A[x(t) - G(t, x_t)] + F(t, x_t) \quad \text{for } t \in [0, a].
\]
These implies that \( x \) is a strict solution of Eq. (2) on \([0, a]\).

4 The solution semigroup in the autonomous case and the linearized stability principle

In this section, we suppose that \( F \) and \( G \) are autonomous. Then Eq. (2) becomes
\[
\begin{cases}
    \frac{d}{dt} [x(t) - G(t, x_t)] = -A[x(t) - G(t, x_t)] + F(t, x_t) \quad \text{for } t \geq 0, \\
x_0 = \varphi \in C_\alpha.
\end{cases}
\] (11)
We can see that the mild solutions of Eq. (11) satisfy the properties of a nonlinear strongly continuous semigroup on \( C_\alpha \) and we prove that this semigroup satisfies the translation property and a Lipschitz property.

For each \( t \geq 0 \), define the nonlinear operator \( U(t) \) on \( C_\alpha \) by
\[
    U(t)(\varphi) = x_t(., \varphi)
\]
where \( x(., \varphi) \) is the unique mild solution of Eq. (11) for the initial condition \( \varphi \in C_\alpha \). One can prove the proposition.

Proposition 4.1. Under the assumption as in the Theorem 2.4, the family \( \{U(t)\}_{t \geq 0} \) is a nonlinear strongly continuous semigroup on \( C_\alpha \). Moreover

(i) \( \{U(t)\}_{t \geq 0} \) satisfies the following translation property, for \( t \geq 0 \) and \( \theta \in [-r, 0] \),
\[
    (U(t)(\varphi))(\theta) = \begin{cases}
        (U(t+\theta)(\varphi))(0), & \text{if } t + \theta \geq 0 \\
        \varphi(t + \theta), & \text{if } t + \theta \leq 0
    \end{cases}
\]
(ii) for all $T > 0$, there are two functions $p, q \in L^\infty([0, T], \mathbb{R}^+)$. such that, for all $\varphi_1, \varphi_2 \in C_\alpha$,

$$
\|U(t)(\varphi_1) - U(t)(\varphi_2)\|_\alpha \leq p(t)e^{q(t)}\|\varphi_1 - \varphi_2\|_\alpha, \quad t \in [0, T].
$$

Proof. Proof of (ii). Let $x^1 := x(., \varphi_1)$, $x^2 := x(., \varphi_2)$, $T > 0$ and $M > 1$ such that $\sup\{\|T(t)\|, t \in [0, T]\} \leq M$. For $t \in [0, T]$, we have

$$
\|U(t)(\varphi_1) - U(t)(\varphi_2)\|_\alpha = \|x^1_t - x^2_t\|_\alpha
= \sup_{-r \leq \theta \leq 0} |x^1(t + \theta) - x^2(t + \theta)|_\alpha
\leq (M + ML_\alpha)\|\varphi_1 - \varphi_2\|_\alpha + L_g \sup_{-r \leq \theta \leq 0} \|x^1_{t+0} - x^2_{t+0}\|_\alpha
+ ML_f \int_0^t \|x^1_s - x^2_s\|_\alpha ds.
$$

Letting $t \in [0, r]$. Then, for $\theta \in [-r, 0]$ such that $t + \theta \geq 0$, we have

$$
\|x^1_{t+0} - x^2_{t+0}\|_\alpha = \sup_{-r \leq \tau \leq 0} |x^1(t + \theta + \tau) - x^2(t + \theta + \tau)|_\alpha
= \sup_{-r + t \leq \tau \leq t + 0} |x^1(\tau) - x^2(\tau)|_\alpha
= \max(\|\varphi_1 - \varphi_2\|_\alpha, \sup_{0 \leq \tau \leq t + 0} \|x^1(\tau) - x^2(\tau)|_\alpha)
\leq \|\varphi_1 - \varphi_2\|_\alpha + \|x^1_t - x^2_t\|_\alpha.
$$

Then,

$$
\|x^1_t - x^2_t\|_\alpha \leq \frac{(M + ML_\alpha + 1)}{1 - L_\alpha}\|\varphi_1 - \varphi_2\|_\alpha + \frac{ML_f}{1 - L_\alpha} \int_0^t \|x^1_s - x^2_s\|_\alpha ds.
$$

Using Gronwall’s lemma, we obtain

$$
\|x^1_t - x^2_t\|_\alpha \leq \frac{(M + ML_\alpha + 1)}{1 - L_\alpha}e^{\frac{ML_f}{1 - L_\alpha}t}\|\varphi_1 - \varphi_2\|_\alpha.
$$

We can repeat the previous argument for $t \in [r, 2r]$, to see that for every $t \in [r, 2r]$,

$$
\|U(t)(\varphi_1) - U(t)(\varphi_2)\|_\alpha \leq \|U(r)\|\|U(t-r)(\varphi_1) - U(t-r)(\varphi_2)\|_\alpha
\leq \frac{(M + ML_\alpha + 1)}{1 - L_\alpha}e^{\frac{ML_f}{1 - L_\alpha}t}\|\varphi_1 - \varphi_2\|_\alpha.
$$

For $t \in [2r, 3r]$

$$
\|U(t)(\varphi_1) - U(t)(\varphi_2)\|_\alpha \leq \|U(2r)\|\|U(t-2r)(\varphi_1) - U(t-2r)(\varphi_2)\|_\alpha
\leq \frac{(M + ML_\alpha + 1)}{1 - L_\alpha}e^{\frac{ML_f}{1 - L_\alpha}t}\|\varphi_1 - \varphi_2\|_\alpha.
$$
Inductively, for \( t \in [nr, (n+1)r] \) with \( n \geq 2 \), we obtain
\[
\|U(t)(\varphi_1) - U(t)(\varphi_2)\| \leq \|U(nr)||U(t-nr)(\varphi_1) - U(t-nr)(\varphi_2)\| \\
\leq \left( \frac{M + ML_g + 1}{1 - L_g} \right)^{n+1} e^{\frac{M\alpha}{L_g}} \|\varphi_1 - \varphi_2\|, 
\]

Consequently, the estimate (12) is true. This ends the proof.

In what follows, we study the stability of an equilibrium of the following autonomous equation:

\[
\begin{cases}
\frac{d}{dt}[D(x_t) - G(x_t)] = -A[D(x_t) - G(x_t)] + F(x_t) & \text{for } t \geq 0, \\
x_0 = \varphi \in C_\alpha, 
\end{cases}
\]

where \( F \) and \( G \) are Lipschitz continuous on \( C_\alpha \) with constants respectively \( L_F \) and \( L_G \) and \( D : C_\alpha \to X_\alpha \) is an operator defined by \( D\varphi = \varphi(0) - D_0\varphi \) with \( D_0 \) a bounded linear operator from \( C_\alpha \) into \( X_\alpha \) such that \( L_G + \|D_0\| < 1 \).

We are now interested by the stability of the equilibriums of Equation (13). By equilibrium, we mean a constant mild solution \( x^* \) of (13). Without loss of generality, we can assume that \( x^* = 0 \) and \( G(0) = F(0) = 0 \):

We need the following assumption.

\textbf{(H6)} \( F \) and \( G \) are Fréchet-differentiable at 0 and \( G'(0) = 0 \).

Let \( L = F'(0) \). Then, the linearized equation of Eq. (13) around the equilibrium 0 is the following:

\[
\begin{cases}
\frac{d}{dt}Dy_t = -ADy_t + L(y_t) & \text{for } t \geq 0, \\
y_0 = \varphi \in C_\alpha, 
\end{cases}
\]

Let \((U(t))_{t \geq 0}\) the nonlinear semigroup associated to Eq. (13) and the linear semigroup \((V(t))_{t \geq 0}\) associated to the linear equation (14) in the same space \( C_\alpha \). Then, we have the following result.

\textbf{Theorem 4.1.} Assume that the conditions (H0), (H2), (H4), (H5) and (H6) hold. Then, for every \( t \geq 0 \) the derivative at zero of \( U(t) \) is \( V(t) \).

The proof of this theorem is based on the following fundamental lemma

\textbf{Lemma 4.2.} Let \( H : C_\alpha \to X_\alpha \) be a continuous function such that there exists \( 0 < \mu_0 < 1 \) satisfying
\[
|H(\varphi_1) - H(\varphi_2)|_\alpha \leq \mu_0 |\varphi_1 - \varphi_2|_\alpha 
\]

Let \( \varphi \in C_\alpha \) and \( h : [0, +\infty[ \to X_\alpha \) be a continuous function. Suppose that there exist continuous functions \( x, y : [-r, +\infty[ \to X_\alpha \) such that

\[
\begin{cases}
x(t) - y(t) = H(x_t) - H(y_t) + h(t), & t \geq 0, \\
x_0 = y_0 = \varphi.
\end{cases}
\]
Then, for each $0 < T \leq r$ we have

$$\|x_t - y_t\|_\alpha \leq \frac{1}{1 - \mu_0} \sup_{0 \leq s \leq t} |h(s)|_\alpha, \quad t \in [0, T].$$

Proof. For $t \geq 0$, we have

$$\|x_t - y_t\|_\alpha = \sup_{-r \leq \theta \leq 0} |x(t + \theta) - y(t + \theta)|_\alpha$$
$$= \sup_{t-r \leq s \leq t} |x(s) - y(s)|_\alpha$$
$$= \sup_{0 \leq s \leq t} |x(s) - y(s)|_\alpha$$
$$\leq \sup_{0 \leq s \leq t} |H(x_s) - H(y_s)|_\alpha + \sup_{0 \leq s \leq t} |h(s)|_\alpha$$
$$\leq \mu_0 \sup_{0 \leq s \leq t} \|x_s - y_s\|_\alpha + \sup_{0 \leq s \leq t} |h(s)|_\alpha$$
$$= \mu_0 \|x_t - y_t\|_\alpha + \sup_{0 \leq s \leq t} |h(s)|_\alpha$$

Then $\|x_t - y_t\|_\alpha \leq \frac{1}{1 - \mu_0} \sup_{0 \leq s \leq t} |h(s)|_\alpha \square$

Proof. (of Theorem 4.1) It suffices to show that for each $\varphi \in C_\alpha$, $t \geq 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|U(t)\varphi - V(t)\varphi\|_\alpha \leq \varepsilon \|\varphi\|_\alpha, \text{ for } \|\varphi\|_\alpha \leq \delta$$

Let $t \geq 0$ be fixed and $\varphi \in C_\alpha$. We have

$$(D - G)(U(t)\varphi) - D(V(t)\varphi)$$
$$= \int_0^T (t - s)[F(U(s)\varphi) - F(V(s)\varphi)]ds - T(t)G(\varphi)$$
$$+ \int_0^T (t - s)[F(V(s)\varphi) - L(V(s)\varphi)]ds$$

Then,

$$(D - G)(U(t)\varphi) - (D - G)(V(t)\varphi)$$
$$= G(V(t)\varphi) - T(t)G(\varphi) + \int_0^T (t - s)[F(U(s)\varphi) - F(V(s)\varphi)]ds$$
$$+ \int_0^T (t - s)[F(V(s)\varphi) - L(V(s)\varphi)]ds$$

Let $x, y : [-r, +\infty[ \rightarrow X_\alpha$ and $h : [0, +\infty[ \rightarrow X_\alpha$ be defined by

$$x(t) = \begin{cases} (U(t)\varphi)(0) & \text{if } t \in [0, +\infty] \\ \varphi(t) & \text{if } t \in [-r, 0] \end{cases}$$
$$y(t) = \begin{cases} (V(t)\varphi)(0) & \text{if } t \in [0, +\infty] \\ \varphi(t) & \text{if } t \in [-r, 0] \end{cases}$$
and
\[ h(t) = G(V(t)\varphi) - T(t)G(\varphi) + \int_0^t T(t-s)[F(U(s)\varphi) - F(V(s)\varphi)]ds \]
\[ + \int_0^t T(t-s)[F(V(s)\varphi) - L(V(s)\varphi)]ds \]

Then,
\[ \begin{cases} (D - G)(x_t) - (D - G)(y_t) = h(t), & t \geq 0, \\
x_0 = y_0 = \varphi. \end{cases} \]

which is equivalent to
\[ \begin{cases} x(t) - y(t) = (D_0 + G)(x_t) - (D_0 + G)(y_t) + h(t), & t \geq 0, \\
x_0 = y_0 = \varphi. \end{cases} \]

Using Lemma 4.2, we obtain
\[ \|x_t - y_t\|_\alpha \leq \frac{1}{1 - (L_G + \|D_0\|)} \sup_{0 \leq s \leq t} |h(s)|_\alpha, \quad t \geq 0. \]

By virtue of the continuous differentiability of $G$ and $F$ at 0, we deduce that for $\varepsilon > 0$, there exists $\delta > 0$ such that
\[ |G(V(t)\varphi) - T(t)G(\varphi)|_\alpha \leq \varepsilon \|\varphi\|_\alpha \text{ for } \|\varphi\|_\alpha \leq \delta, \]

and
\[ M_\alpha \int_0^t |F(V(s)\varphi) - L(V(s)\varphi)|\frac{ds}{(t-s)^\alpha} \leq \varepsilon \|\varphi\|_\alpha \text{ for } \|\varphi\|_\alpha \leq \delta. \]

Then, for $\|\varphi\|_\alpha \leq \delta,$
\[ |h(t)|_\alpha \leq 2\varepsilon \|\varphi\|_\alpha + M_\alpha L_F \int_0^t \|U(s)\varphi - V(s)\varphi\|_\alpha \frac{ds}{(t-s)^\alpha}. \]

Since for $s \in [0, t]$ and $t \in [0, r],$
\[ \|U(s)\varphi - V(s)\varphi\|_\alpha = \sup_{-r \leq \tau \leq 0} |x_\tau(\varphi) - y_\tau(\varphi)|_\alpha \]
\[ = \sup_{-r \leq \tau \leq t} |x(\tau) - y(\tau)|_\alpha \]
\[ = \sup_{0 \leq \tau \leq t} |x(\tau) - y(\tau)|_\alpha \]
\[ \leq \sup_{0 \leq \tau \leq t} |x(\tau) - y(\tau)|_\alpha \]
\[ = \|U(t)\varphi - V(t)\varphi\|_\alpha. \]

Then for $t \in [0, r]$ fixed
\[ \|U(t)\varphi - V(t)\varphi\|_\alpha \leq \frac{2\varepsilon \|\varphi\|_\alpha}{1 - (L_G + \|D_0\|)} + \frac{M_\alpha L_F}{1 - (L_G + \|D_0\|)} \int_0^t \|U(s)\varphi - V(s)\varphi\|_\alpha \frac{ds}{(t-s)^\alpha}. \]
Using Gronwall’s lemma, we obtain
\[ \|U(t)φ - V(t)φ\|_α ≤ \frac{2ε\|φ\|_α}{1 - (\|L_φ + \|D_0\|)} \exp\left(\frac{M_αL_φt^{1-α}}{1 - (\|L_φ + \|D_0\|)(1-α)}\right) \]
for \(\|φ\|_α ≤ δ\). We conclude that \(U(t)\) is differentiable at 0, for each \(t \in [0, T]\) and \(D_φU(t)(0) = V(t)\).

Now, suppose that \(t ∈ [T, 2T]\) fixed. It follows that, for max \(\{\|φ\|_α, \|U(t-T)(φ)\|_α\} ≤ δ_0\), where \(δ_0 > 0\) is small enough
\[ \|U(t)φ - V(t)φ\|_α ≤ \|U(T)U(t-T)(φ) - V(T)U(t-T)(φ)\|_α \]
\[ + \|V(T)\|U(t-T)(φ) - V(t-T)(φ)\|_α \]
\[ ≤ ε\|φ\|_α. \]

By steps, we conclude that \(U(t)\) is differentiable at 0, for each \(t ≥ 0\) and \(D_φU(t)(0) = V(t)\).

\[ \textbf{Theorem 4.2.} \] Under the assumption as in the Theorem 4.1, if the zero equilibrium of \((V(t))_{t ≥ 0}\) is exponentially stable, then the zero equilibrium of \((U(t))_{t ≥ 0}\) is locally exponentially stable, in the sense that there exist \(δ > 0, µ > 0\) and \(k ≥ 1\) such that
\[ \|U(t)φ\|_α ≤ ke^{-µt}\|φ\|_α \quad \text{for } t ≥ 0 \text{ and } φ ∈ C_α \text{ with } \|φ\|_α ≤ δ. \]

Moreover, if \(C_α\) can be decomposed as \(C_α = H_1 ⊕ H_2\) where \(H_1\) are \(V\)-invariant subspaces of \(C_α\), \(H_1\) is finite-dimensional and with
\[ \omega = \lim_{n→∞} \frac{1}{h} \log\|V(h)/H_2\|_α, \]
we have
\[ \inf\{|λ| : λ ∈ σ(V(t)/H_1)| > e^{ωt}, \]
then, the zero equilibrium of \((U(t))_{t ≥ 0}\) is not stable, in the sense that there exist \(ε > 0\) and a sequence \((φ_n)_n\) converging to 0 and a sequence \((t_n)_n\) of positive real numbers such that \(\|U(t_n)φ_n\|_α > ε\).

The proof of this theorem is based on Proposition 4.1, Theorem 4.1 and the following result.

\[ \textbf{Theorem 4.3.} \] (Desch and Schappacher [13]). Let \((V(t))_{t ≥ 0}\) be a nonlinear strongly continuous semigroup on a subset \(Ω\) of a Banach space \(Z\). Assume that \(x_0 ∈ Ω\) is an equilibrium of \((V(t))_{t ≥ 0}\) such that \(V(t)\) is Fréchet-differentiable at \(x_0\) for each \(t ≥ 0\), with \(W(t)\) the derivative at \(x_0\) of \(V(t)\), \(t ≥ 0\). Then, \((W(t))_{t ≥ 0}\) is a strongly continuous semigroup of bounded linear operators on \(Z\) and, if the zero equilibrium of \((W(t))_{t ≥ 0}\) is exponentially stable, then the equilibrium \(x_0\) of \((V(t))_{t ≥ 0}\) is locally exponentially stable. Moreover, if \(Z\) can be decomposed as \(Z = Z_1 ⊕ Z_2\) where \(Z_1\) are \(W\)-invariant subspaces of \(Z\) and \(Z_1\) is finite-dimensional and with
\[ \omega = \lim_{h→∞} \frac{1}{h} \log\|W(h)/Z_2\|, \]
we have
\[ \inf\{|\lambda| : \lambda \in \sigma(W(t)/Z_{1})\} > e^{\alpha t}, \]
then, the zero equilibrium \( x_0 \) of \( (V(t))_{t \geq 0} \) is not stable, in the sense that there exist \( \varepsilon > 0 \) and a sequence \( \{x_n\}_n \) converging to \( x_0 \) and a sequence \( \{t_n\}_n \) of positive real numbers such that \( \|V(t_n)x_n - x_0\| > \varepsilon. \)

In the following, we will concentrate our study on the linear equation (14). Let \( (A_{V}, D(A_{V})) \) be the generator of the semigroup \( (V(t))_{t \geq 0} \) on \( C_{\alpha} \). We have the result

**Theorem 4.4.** \([4]\) Assume that the conditions (H0), (H2), (H4), (H5) and (H6) hold. Then, the operator \( (A_{V}, D(A_{V})) \) is given by

\[
\begin{cases}
D(A_{V}) = \{ \varphi \in C_{\alpha}, \varphi' \in C_{\alpha}, D(\varphi) \in D(A) \text{ and } D(\varphi') = -AD(\varphi) + L(\varphi) \}, \\
A_{V}\varphi = \varphi', \quad \varphi \in D(A_{V}).
\end{cases}
\]

Let \( C \) be the space of continuous functions from \([-r,0]\) into \( X \) provided with the uniform norm topology and let \( C_{D} = \{ \varphi \in C : D(\varphi) = 0 \} \).

**Definition 4.3.** \([22]\) \( D \) is said to be stable if the zero solution of the difference equation

\[
\begin{cases}
D(y_t) = 0, \quad t \geq 0, \\
y_0 = \varphi \in C_{D},
\end{cases}
\]

is exponentially stable.

**Lemma 4.4.** \([4]\) If \( D \) is stable, then there exist positive constants \( a, b, c \) and \( d \) such that for any \( \varepsilon \in ]0,r[ \) sufficiently small and any continuous function \( h \) from \([0, +\infty[\) into \( X \), the solution \( v \) of the equation

\[
D(v_t) = h(t), \quad t \geq 0,
\]

satisfies the inequality

\[
\|v_t\| \leq e^{-a(t-\varepsilon)} \left[ b\|v_0\| + c \sup_{0 \leq s \leq \varepsilon} |h(s)| \right] + d \sup_{\max(\varepsilon, t-r) \leq s \leq t} |h(s)|, \quad t \geq \varepsilon. \quad (15)
\]

The estimate (15) is very interesting because, if \( |h(s)| \) is bounded on \([0, +\infty[\), then the ultimate bound on \( v_t \) as \( t \to +\infty \) is determined by the bound on \( |h(s)| \) for \( s \) in the delay interval \([t-r, t]\) as \( t \to +\infty \).

**Proposition 4.5.** \([20]\) Let \( D(\varphi) = \sum_{k=0}^{p} a_k \varphi(-r_k) \). Then, \( D \) is stable iff \( \sum_{k=0}^{p} |a_k| < 1 \).

In the sequel, we assume that
(H7) The operator $D$ is stable.

**Theorem 4.5.** [4] Assume that (H0), (H1), (H2), (H4), (H5) and (H7) hold. Then the semigroup $(U(t))_{t \geq 0}$ can be decomposed as follows

$$U(t) = U_1(t) + U_2(t) \quad \text{for } t \geq 0,$$

where $U_1(t)$ is an exponentially stable semigroup on $C_\alpha$ and $U_2(t)$ is compact on $C_\alpha$ for every $t > 0$.

Let $(Y, \|\|)$ be a Banach space. For a bounded linear operator $B$ in $Y$, we define

$$\|B\|_{\text{ess}} := \inf \{c > 0 : \chi(B(H)) \leq c\chi(H), \, \text{for every bounded set } H \text{ of } Y\},$$

where $\chi(,)$ denotes the measure of noncompactness in $Y$. The essential growth bound of $(V(t))_{t \geq 0}$ in $C_\alpha$ is given by

$$\omega_{\text{ess}}(V) := \inf_{t > 0} \frac{1}{t} \log \|V(t)\|_{\text{ess}}.$$

It follows from Theorem 4.5 that

$$\omega_{\text{ess}}(V) < 0.$$

Let

$$\omega_0(V) := \inf_{t > 0} \frac{1}{t} \log \|V(t)\|_\alpha$$

be the growth bound of $(V(t))_{t \geq 0}$ in $C_\alpha$. Then, it is well known (see [14]) that

$$\omega_0(V) = \max(\omega_{\text{ess}}(V), s'(A_V)),$$

where

$$s'(A_V) = \sup \{\Re \lambda : \lambda \in \sigma(A_V) \setminus \sigma_{\text{ess}}(A_V)\}$$

and $\sigma_{\text{ess}}(A_V)$ is the essential spectrum of $A_V$. Consequently, the stability of $(V(t))_{t \geq 0}$ is completely determined by $s'(A_V)$. Note that $\sigma(A_V) \setminus \sigma_{\text{ess}}(A_V)$ contains a finite number of eigenvalues of $A_V$.

We say that $\lambda \in \mathbb{C}$ is a characteristic value of Equation (14) if there exists a nonzero $x \in D(\Delta(\lambda)) \setminus \{0\}$ such that $\Delta(\lambda)x = 0$, where $\Delta(\lambda)$ is defined by

$$\Delta(\lambda) := \lambda D(e^{\lambda t}) + AD(e^{\lambda t}) - L(e^{\lambda t})$$

and the domain $D(\Delta(\lambda))$ is given by

$$D(\Delta(\lambda)) := \{x \in X_\alpha : D(e^{\lambda t}x) \in D(A) \text{ and } AD(e^{\lambda t}x) - L(e^{\lambda t}x) \in X_\alpha\}.$$ 

Consequently, we deduce the following theorem.

**Theorem 4.6.** [4] Assume that (H0), (H1), (H2), (H4), (H5), (H6) and (H7) hold. Then, the following assertions hold
(i) \( \lambda \) is an eigenvalue of \( A_V \) iff \( \lambda \) is a characteristic value of Equation (14).

(ii) If \( s'(A_V) < 0 \), then \( (V(t))_{t \geq 0} \) is exponentially stable and consequently, the zero equilibrium of \( (U(t))_{t \geq 0} \) is locally exponentially stable.

(iii) If \( s'(A_V) = 0 \), then there exists \( \varphi \in C_\alpha, \varphi \neq 0 \), such that \( \|V(t)\varphi\|_\alpha = \|\varphi\|_\alpha \), for \( t \geq 0 \).

(iv) If \( s'(A_V) > 0 \), then there exists \( \varphi \in C_\alpha \) such that \( \|V(t)\varphi\|_\alpha \to +\infty \) as \( t \to +\infty \) and consequently, the zero equilibrium of \( (U(t))_{t \geq 0} \) is unstable.

(v) Assume that \( s'(A_V) \leq 0 \) and let \( s_0(A_V) := \{ \lambda \in \text{P}_0(A_V) : \text{Re}\lambda = 0 \} \). If each \( \lambda \) in \( s_0(A_V) \) is a pole of order 1 of the resolvent operator of \( A_V \), then \( (V(t))_{t \geq 0} \) is stable in the sense that there exists a positive constant \( M \) such that \( \|V(t)\|_\alpha \leq M \), for all \( t \geq 0 \).

5 Example

To apply our theoretical results, we consider the following model of partial differential equation with delay

\[
\begin{cases}
\frac{\partial}{\partial t} [v(t,x) - qv(t-r,t,x) + g(qv(t-r,t,x))] = \frac{\partial^2}{\partial x^2} [v(t,x) - qv(t-r,t,x)] + g(v(t-r,t,x)
\quad \frac{\partial}{\partial x} [v(t,x) - qv(t-r,t,x)])] \\
\quad \text{for } t \geq 0 \text{ and } x \in [0,\pi], \\
v(t,0) - qv(t-r,0) = v(t,\pi) - qv(t-r,\pi) = 0 \quad \text{for } t \geq 0, \\
v(\theta,x) = v_0(\theta,x) \quad \text{for } -\pi \leq \theta \leq 0 \text{ and } x \in [0,\pi],
\end{cases}
\]

where \( q, r \) are positive constants, \( u_\theta \in C([-r,0] \times [0,\pi]; \mathbb{R}) \) and \( f, g \) are Lipschitz continuous functions. Let \( X := L^2([0,\pi]; \mathbb{R}) \) equipped with the \( L^2 \)-norm \( \| \cdot \|_2 \). Consider the operator \( A : D(A) \subset X \to X \) defined by \( Ay = -y'' \) with domain \( D(A) = H^2(0,\pi) \cap H^1_0(0,\pi) \). The spectrum \( \sigma(-A) \) of \( -A \) is equal to the point spectrum \( \sigma_p(-A) \) and is given by \( \sigma(-A) = \sigma_p(-A) = \{-n^2 : n \geq 1 \} \) and the associated eigenfunctions \( \{e_n\}_{n \geq 1} \) are given by \( e_n(s) = \sqrt{\frac{2}{\pi}} \sin ns, s \in [0,\pi] \). Then

\[
A_{\alpha} y = \sum_{n=1}^{\infty} n^2 (y,e_n)e_n, \quad y \in D(A').
\]

For each \( y \in D(A'_{\alpha}) := \{ y \in X : \sum_{n=1}^{\infty} n(y,e_n)e_n \in X \} \) the operator \( A'_{\alpha} \) is given by \( A'_{\alpha} y = \sum_{n=1}^{\infty} n(y,e_n)e_n \).

It is well known that \( -A \) is the infinitesimal generator of an analytic semigroup \( (T(t))_{t \geq 0} \) on
X given by \( T(t)x = \sum_{n=1}^{\infty} e^{-n^2t} e_n e_n, \ x \in X \). It follows that \( (T(t))_{t \geq 0} \) is a compact semigroup on X and \( 0 \in \rho(A) \). This implies that the Assumption \((H0)\) and \((H1)\) are satisfied.

**Lemma 5.1.** \[26\] If \( Y \in D(A^\frac{1}{2}) \), then \( Y \) is absolutely continuous, \( Y' \in X \) and \( \|Y'\| = \|A^\frac{1}{2}Y\| \).

Let \( G : C^1 \to X \) be defined by

\[ G(\phi)(x) = q\phi(-r)(x) - g\left(\frac{\partial}{\partial x} \phi(-r)(x)\right) \quad \text{for} \ \phi \in C^1 \ \text{and} \ x \in [0, \pi], \]

and \( F : C^1 \to X \) be defined by

\[ F(\phi)(x) = f\left(\phi(-r)(x), \frac{\partial}{\partial x} [q\phi(0)(x) - q\phi(-r)(x)]\right) \quad \text{for} \ \phi \in C^1 \ \text{and} \ x \in [0, \pi]. \]

**Lemma 5.2.** \[27\] \( F \) and \( G \) are Lipschitz continuous from \( C^1 \) into \( X \).

Let \( x(t) = v(t, .) \) for \( t \geq 0 \) and \( \theta(t) = v_0(\theta, .) \) for \( \theta \in [-r, 0] \). Then, Eq. \((16)\) takes the following abstract form

\[
\begin{cases}
\frac{d}{dt} (x(t) - G(t, x_1)) = -A(x(t) - G(t, x_1)) + F(t, x_1) & \text{for} \ t \geq 0, \\
x_0 = \phi.
\end{cases}
\]

Consequently, we have the existence and uniqueness of the mild solution of Eq. \((16)\). Let \( v_0 \in C^1 \) such that

(a) \( v_0(\theta, .) - qv_0(-r, .) + g\left(\frac{\partial}{\partial x} v_0(-r, .)\right) \in H^2(0, \pi) \cap H^1(0, \pi) \) and \( \frac{\partial}{\partial \theta} v_0 \in C^1 \),

(b) \( \frac{\partial}{\partial \theta} v_0(0, x) - q \frac{\partial}{\partial x} v_0(-r, x) + g\left(\frac{\partial}{\partial x} v_0(-r, x)\right) \frac{\partial^2}{\partial x \partial \theta} v_0(-r, x) \)

\[ = -A \left(v_0(0, x) - qv_0(-r, x) + g\left(\frac{\partial}{\partial x} v_0(-r, x)\right)\right) \]

\[ + f\left(v_0(-r, x), \frac{\partial}{\partial x} [v_0(0, x) - qv_0(-r, x)]\right) \quad \text{for} \ x \in [0, \pi]. \]

We deduce that all assumptions of Theorem \[5.3\] are satisfied. Hence every mild solution of Eq. \((16)\) is a strict solution.

In the sequel, we assume that \( 0 < q < 1 \): This means that the operator \( D \) is stable. We also assume that \( f \) and \( g \) are continuously differentiable and \( f(0, 0) = 0, g(0) = 0 \) and \( g'(0) = 0 \). Which implies that zero is a solution of \((14)\) and the linearized equation at zero of Equation \((14)\) has the following form

\[
\begin{cases}
\frac{\partial}{\partial t} [v(t, x) - qv(t - r, x)] = a^2 \frac{\partial^2}{\partial x^2} [v(t, x) - qv(t - r, x)] \\
+ av(t - r, x) + b \frac{\partial}{\partial x} [v(t, x) - qv(t - r, x)] & \text{for} \ t \geq 0 \ \text{and} \ x \in [0, \pi], \\
v(t, 0) - qv(t, 0) = v(t, \pi) - qv(t - r, \pi) = 0 & \text{for} \ t \geq 0, \\
v(0, x) = v_0(\theta, x) & \text{for} \ -r \leq \theta \leq 0 \ \text{and} \ x \in [0, \pi],
\end{cases}
\]

\[ (18) \]
We obtained a region of stability of Equation (18) as a function of parameters $a$, $b$ and $q$.

**Lemma 5.3.** [4] The spectrum $\sigma(\tilde{A})$ of the operator $\tilde{A} = \frac{\partial^2}{\partial x^2} + b \frac{\partial}{\partial x}$ is equal to the point spectrum $\sigma_p(\tilde{A})$ and is given by $\{-n^2 - \frac{b^2}{4} : n \geq 1\}$.

**Theorem 5.1.** Suppose that $a < 0$ and $1 + \frac{b^2}{4} + \frac{a}{q} \geq 0$.

Then, for every $r > 0$, all characteristic values of Eq. (18) have negative real parts.

**Proof.** Suppose that $a < 0$. Then, the characteristic values of Eq. (18) are determined by the expression

$$\lambda - \frac{ae^{-\lambda r}}{1 - qe^{-\lambda r}} = -n^2 - \frac{b^2}{4}, \quad n \geq 1. \quad (19)$$

Let $K_n = n^2 + \frac{b^2}{4}$, $n \geq 1$. Then, Eq. (19) becomes

$$e^{\lambda r}(\lambda + k_n) = \lambda q + K_n q + a.$$ 

This implies that

$$e^{2\text{Re}(\lambda)r}((\text{Re}(\lambda) + K_n)^2 + (\text{Im}(\lambda))^2) \geq q^2 ((\text{Re}(\lambda) + K_n + \frac{a}{q}) + (\text{Im}(\lambda))^2).$$

On the other hand, under the conditions

$$a < 0 \text{ and } 1 + \frac{b^2}{4} + \frac{a}{q} \geq 0,$$

we have, for all $n \geq 1$ and $\lambda \in \mathbb{C}$,

$$\text{Re}(\lambda) + K_n > \text{Re}(\lambda) + K_n + \frac{a}{q} \geq \text{Re}(\lambda) + 1 + \frac{b^2}{4} + \frac{a}{q} \geq \text{Re}(\lambda).$$

Then, if we assume that $\text{Re}(\lambda) \geq 0$, we obtain that

$$e^{2\text{Re}(\lambda)r} < q^2,$$

which is a contradiction. Then, $\text{Re}(\lambda) < 0$. $\square$

Remark that the stability result is independent of the delay. Finally, as an immediate consequence of the last theorem, we have the local stability of the zero equilibrium of Equation (16).

**Proposition 5.4.** Under the same assumptions as in Theorem 5.1, zero equilibrium of Equation (16) is locally exponentially stable.

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References


Existence and stability of almost periodic solutions to impulsive stochastic differential equations

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ABSTRACT
This paper introduces the concept of square-mean piecewise almost periodic for impulsive stochastic processes. The existence of square-mean piecewise almost periodic solutions for linear and nonlinear impulsive stochastic differential equations is established by using the theory of the semigroups of the operators and Schauder fixed point theorem. The stability of the square-mean piecewise almost periodic solutions for nonlinear impulsive stochastic differential equations is investigated.

RESUMEN
Este artículo introduce el concepto de periodicidad cuadrática media por tramos casi periódica para procesos estocásticos impulsivos. La existencia de soluciones de media cuadrática casi periódicas para ecuaciones diferenciales estocásticas impulsivas lineales y no lineales se establece usando la teoría de semigrupos de los operadores y el teorema de punto fijo de Schauder. Se estudia la estabilidad de las soluciones de media cuadrática por tramos casi periódica para ecuaciones diferenciales estocásticas impulsivas no lineales.

Keywords and Phrases: Square-mean piecewise almost periodic; impulsive stochastic differential equation; the semigroups of the operators; Schauder fixed point theorem; stability

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1 Introduction

In recent years, stochastic differential systems have been extensively studied since stochastic modeling plays an important role in physics, engineering, finance, social science and so on. Qualitative properties such as existence, uniqueness and stability for stochastic differential systems have attracted more and more researchers’ attention. The existence of periodic, almost periodic(automorphic), asymptotically almost periodic, pseudo almost periodic(automorphic) solutions for stochastic differential equations was obtained. We refer the reader to [14, 6, 7, 17, 16, 10, 8, 1, 11] and references therein.

On the other hand, impulsive phenomenon arises from many different real processes and phenomena which appeared in physics, chemical technology, population dynamics, biotechnology, medicine and economics. There has been a significant development in the theory of impulsive differential equations. For example, the existence of almost periodic (mild) solutions of abstract impulsive differential equations have been considered in [23, 24, 25, 4, 18, 19].

In [26], the authors combined the two directions and derived firstly some sufficient conditions for the existence and uniqueness of almost periodic solutions for a class of impulsive stochastic differential equations with delay. However, these above results quoted concern the case where the activation functions satisfy Lipschitz conditions. There are few authors have considered the problem of almost periodic solutions of impulsive stochastic differential equations without Lipschitz activation functions. On the basis of this, this article is devoted to the discussion of this problem.

Moreover, the stability analysis on impulsive stochastic differential equations has been an important research topic (see [20, 22, 27]). While, because the mild solutions don’t have stochastic differentials, Ito’s formula fails to deal with the stability of mild solution to stochastic differential equations (see [20, 9, 15]). In [9], the authors gave some properties of the stochastic convolution which ensure the exponential stability of mild solutions.

Motivated by the above discussion, we investigate the existence and stability of almost periodic solutions for impulsive stochastic differential equations. The paper is organized as follows, in Section 2 we recall some definitions, the related notations and some useful lemmas. In Sections 3 and 4, we present some criteria ensuring the existence of almost periodic solutions to some linear and nonlinear impulsive stochastic differential equations, respectively. In Section 5, we discuss the stability of almost periodic solutions to some impulsive stochastic differential equations.

2 Preliminaries

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of nonnegative real numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{Z}^+$ denotes the set of nonnegative integers. $(\mathcal{H}, || \cdot ||)$ is assumed to be a real and separable Hilbert space. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $L^2(\mathbb{P}, \mathcal{H})$ be a space of the $\mathcal{H}$-valued random variables $x$ such that $E||x||^2 = \int_{\Omega} ||x||^2 d\mathbb{P} < \infty$. 
L^2(P,H) is a Hilbert space equipped with the norm \( \|x\|_2 = (\int_\Omega \|x\|^2 \, dP)^{1/2} \).

**Definition 2.1.** A stochastic process \( x: \mathbb{R}^+ \to L^2(P,H) \) is said to be stochastically bounded if there exists \( M > 0 \) such that \( E\|x(t)\|^2 \leq M \) for all \( t \in \mathbb{R}^+ \).

**Definition 2.2.** A stochastic process \( x: \mathbb{R}^+ \to L^2(P,H) \) is said to be stochastically continuous in \( s \in \mathbb{R}^+ \), if \( \lim_{t \to s} E\|x(t) - x(s)\|^2 = 0 \).

Let \( T \) be the set consisting of all real sequences \( \{t_i\}_{i \in \mathbb{Z}^+} \) such that \( \gamma = \inf_{i \in \mathbb{Z}^+} (t_{i+1} - t_i) > 0 \), \( t_0 = 0 \) and \( \lim_{i \to \infty} t_i = \infty \). \( x(t_i^+) \) and \( x(t_i^-) \) represent the right and left limits of \( x(t) \) at \( t_i, i \in \mathbb{Z}^+ \), respectively. For \( \{t_i\}_{i \in \mathbb{Z}^+} \in T \), let \( PC(\mathbb{R}^+, L^2(P,H)) \) be the space consisting of all stochastically bounded functions \( \phi: \mathbb{R}^+ \to L^2(P,H) \) such that \( \phi(\cdot) \) is stochastically continuous at \( t \) for any \( t \notin \{t_i\}_{i \in \mathbb{Z}^+} \) and \( \phi(t_i) = \phi(t_i^-) \) for all \( i \in \mathbb{Z}^+ \); let \( PC(\mathbb{R}^+ \times L^2(P,H), L^2(P,H)) \) be the space formed by all stochastic processes \( \phi: \mathbb{R}^+ \times L^2(P,H) \to L^2(P,H) \) such that for any \( x \in L^2(P,H) \), \( \phi(t,x) \) is stochastically continuous at \( t \) for any \( t \notin \{t_i\}_{i \in \mathbb{Z}^+} \) and \( \phi(t_i,x) = \phi(t_i^-), x \) for all \( i \in \mathbb{Z}^+ \) and for any \( t \in \mathbb{R}^+ \), \( \phi(t,\cdot) \) is stochastically continuous at \( x \in L^2(P,H) \).

**Definition 2.3.** For \( \{t_i\}_{i \in \mathbb{Z}^+} \in T \), the function \( \phi \in PC(\mathbb{R}^+, L^2(P,H)) \) is said to be square-mean piecewise almost periodic if the following conditions are fulfilled:

1. \( \{t_i\}_{i \in \mathbb{Z}^+} \) is equipotentially almost periodic, that is, for any \( \epsilon > 0 \), there exists a relatively dense set \( Q_{\epsilon} \) of \( \mathbb{R} \) such that for each \( \tau \in Q_{\epsilon} \) there is an integer \( q \in \mathbb{Z} \) such that \( |t_{i+q} - t_i - \tau| < \epsilon \) for all \( i \in \mathbb{Z}^+ \).

2. For any \( \epsilon > 0 \), there exists a positive number \( \delta = \delta(\epsilon) \) such that if the points \( t' \) and \( t'' \) belong to the same interval of continuity of \( \phi \) and \( |t' - t''| < \delta \), then \( E\|\phi(t') - \phi(t'')\|^2 < \epsilon \).

3. For every \( \epsilon > 0 \), there exists a relatively dense set \( \Omega(\epsilon) \) in \( \mathbb{R} \) such that if \( \tau \in \Omega(\epsilon) \), then

\[
E\|\phi(t + \tau) - \phi(t)\|^2 < \epsilon
\]

for all \( t \in \mathbb{R}^+ \) satisfying the condition \( |t - t_i| > \epsilon \), \( i \in \mathbb{Z}^+ \). The number \( \tau \) is called \( \epsilon \)-translation number of \( \phi \).

We denote by \( AP_T(\mathbb{R}^+, L^2(P,H)) \) the collection of all the square-mean piecewise almost periodic processes, it thus is a Banach space with the norm \( \|x\|_\infty = \sup_{t \in \mathbb{R}^+} \|x(t)\|_2 = \sup_{t \in \mathbb{R}^+} (E\|x(t)\|^2)^{1/2} \) for \( x \in AP_T(\mathbb{R}^+, L^2(P,H)) \).

**Lemma 2.4.** Let \( f \in AP_T(\mathbb{R}^+, L^2(P,H)) \), then, \( R(f) \), the range of \( f \) is a relatively compact set of \( L^2(P,H) \).

Refer to [18] for the detailed proof of Lemma 2.4.

**Definition 2.5.** For \( \{t_i\}_{i \in \mathbb{Z}^+} \in T \), the function \( f(t,x) \in PC(\mathbb{R}^+ \times L^2(P,H), L^2(P,H)) \) is said to be square-mean piecewise almost periodic in \( t \in \mathbb{R}^+ \) and uniform on compact subset of \( L^2(P,H) \) if for every \( \epsilon > 0 \) and every compact subset \( K \subseteq L^2(P,H) \), there exists a relatively dense subset \( \Omega \) of \( \mathbb{R} \) such that

\[
E\|f(t + \tau,x) - f(t,x)\|^2 < \epsilon,
\]
for all $x \in K$, $\tau \in \Omega$, $t \in \mathbb{R}^+$ satisfying $|t - t_i| > \varepsilon$. The collection of all such processes is denoted by $AP_T(\mathbb{R}^+ \times L^2(P,H), L^2(P,H))$.

**Lemma 2.6.** Suppose that $f(t,x) \in AP_T(\mathbb{R}^+ \times L^2(P,H), L^2(P,H))$ and $f(t,\cdot)$ is uniformly continuous on each compact subset $K \subseteq L^2(P,H)$ uniformly for $t \in \mathbb{R}$. That is, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x,y \in K$ and $E|\|x - y\|^2 < \delta$ implies that $E|\|f(t,x) - f(t,y)\|^2 < \varepsilon$ for all $t \in \mathbb{R}$. Then $f(\cdot,x(\cdot)) \in AP_T(\mathbb{R}^+, L^2(P,H))$ for any $x \in AP_T(\mathbb{R}^+, L^2(P,H))$.

**Proof.** Since $x \in AP_T(\mathbb{R}^+, L^2(P,H))$, by Lemma 2.4, $R(x)$ is a relatively compact subset of $L^2(P,H)$. Because $f(t,\cdot)$ is uniformly continuous on each compact subset $K \subseteq L^2(P,H)$ uniformly for $t \in \mathbb{R}$, then for any $\varepsilon > 0$, there exists number $\delta : 0 < \delta \leq \frac{\varepsilon}{4}$, such that

$$E|\|f(t,x_1) - f(t,x_2)\|^2 < \frac{\varepsilon}{4},$$

where $x_1,x_2 \in R(x)$ and $E|\|x_1 - x_2\|^2 < \delta$, $t \in \mathbb{R}$. By square-mean piecewise almost periodic of $f$ and $x$, there exists a relatively set $\Omega$ of $\mathbb{R}$ such that the following conditions hold:

$$E|\|f(t + \tau,x_0) - f(t,x_0)\|^2 < \frac{\varepsilon}{4},$$

$$E|\|x(t + \tau) - x(t)\|^2 < \frac{\varepsilon}{4},$$

for every $x_0 \in R(x)$ and $t \in \mathbb{R}^+$, $|t - t_i| > \varepsilon$, $i \in Z^+$, $\tau \in \Omega$. Note that $(a + b)^2 \leq 2(a^2 + b^2)$ and

$$E|\|f(t + \tau,x(t + \tau)) - f(t,x(t))\|^2$$

$$\leq 2E|\|f(t + \tau,x(t + \tau)) - f(t + \tau,x(t))\|^2 + 2E|\|f(t + \tau,x(t)) - f(t,x(t))\|^2.$$

Combining (1), (2) and (3), it follows that

$$E|\|f(t + \tau,x(t + \tau)) - f(t,x(t))\|^2 < \varepsilon,$$

$t \in \mathbb{R}^+$, $|t - t_i| > \varepsilon$, $i \in Z^+$, $\tau \in \Omega$.

The proof is complete. \qed

We obtain the following corollary as an immediate consequence of Lemma 2.6.

**Corollary 2.7.** Let $f(t,x) \in AP_T(\mathbb{R}^+ \times L^2(P,H), L^2(P,H))$ and $f$ is Lipschitz, i.e., there is a number $L > 0$ such that

$$E|\|f(t,x) - f(t,y)\|^2 \leq LE|\|x - y\|^2,$$

for all $t \in \mathbb{R}^+$ and $x,y \in L^2(P,H)$, if for any $x \in AP_T(\mathbb{R}^+, L^2(P,H))$, then $f(\cdot, x(\cdot)) \in AP_T(\mathbb{R}^+, L^2(P,H))$.

**Definition 2.8.** A sequence $x : Z^+ \rightarrow L^2(P,H)$ is called a square-mean almost periodic sequence if the $\varepsilon$-translation set of $x$

$$\mathcal{T}(x; \varepsilon) = \{\tau \in Z : E|\|x(n + \tau) - x(t)\|^2 < \varepsilon, \text{ for all } n \in Z^+\}$$

is a relatively dense set in $Z$ for all $\varepsilon > 0$.

The collection of all square-mean almost periodic sequences $x : Z^+ \rightarrow L^2(P,H)$ will be denoted by $AP_T(Z^+, L^2(P,H))$. 


Lemma 2.12. Here we omit the proofs.

Lemma 35 in [23], respectively, and one may refer to [23, 18, 19, 26, 213, 12] for more details.

Remark 2.9. If \( x(n) \in AP_{1}(Z^{+}, L^{2}(P, H)) \), then \( \{x(n) : n \in Z^{+}\} \) is stochastically bounded, that is, \( \sup_{n \in Z^{+}} E|x(n)|^{2} < \infty \).

In order to obtain our main results, we introduce the following lemmas.

Let \( h : R^{+} \to R \) be a continuous function such that \( h(t) \geq 1 \) for all \( t \in R^{+} \) and \( h(t) \to \infty \) as \( t \to \infty \). We consider the space

\[
(\text{PC})_{h}^{2}(R^{+}, L^{2}(P, H)) = \left\{ u \in \text{PC}(R^{+}, L^{2}(P, H)) : \lim_{t \to \infty} \frac{E|u(t)|^{2}}{h(t)} = 0 \right\}.
\]

Endowed with the norm \( ||u||_{h} = \sup_{t \in R^{+}} \frac{E|u(t)|^{2}}{h(t)} \), it is a Banach space.

Lemma 2.10. A set \( B \subseteq (\text{PC})_{h}^{2}(R^{+}, L^{2}(P, H)) \) is a relatively compact set if and only if

1. \( \lim_{t \to \infty} \frac{E|u(t)|^{2}}{h(t)} = 0 \) uniformly for \( x \in B \).
2. \( B(t) = \{x(t) : x \in B\} \) is relatively compact in \( L^{2}(P, H) \) for every \( t \in R^{+} \).
3. The set \( B \) is equicontinuous on each interval \( (t_{i}, t_{i+1}) (i \in Z^{+}) \).

Lemma 2.11. Assume that \( f \in AP_{1}(R^{+}, L^{2}(P, H)) \), the sequence \( \{x_{i} : i \in Z^{+}\} \) is almost periodic in \( L^{2}(P, H) \) and \( \{t_{i}\} \), \( i \in Z^{+} \), is equiperiodically almost periodic. Then for each \( \epsilon > 0 \) there are relatively dense sets \( \Omega_{\epsilon, f, x_{i}} \) of \( R \) and \( Q_{\epsilon, f, x_{i}} \) of \( Z \) such that the following conditions hold:

1. \( E|f(t + \tau) - f(t)|^{2} < \epsilon \) for all \( t \in R^{+} \), \( |t - t_{i}| > \epsilon \), \( \tau \in \Omega_{\epsilon, f, x_{i}} \) and \( i \in Z^{+} \).
2. \( E|x_{i+q} - x_{i}|^{2} < \epsilon \) for all \( q \in Q_{\epsilon, f, x_{i}} \) and \( i \in Z^{+} \).
3. For every \( \tau \in \Omega_{\epsilon, f, x_{i}} \), there exists at least one number \( q \in Q_{\epsilon, f, x_{i}} \) such that

\[
|t_{i}^{q} - \tau| < \epsilon, \quad i \in Z^{+}.
\]

Lemma 2.10 and Lemma 2.11 are stochastic generalized versions of Lemma 4.1 in [12] and Lemma 35 in [23], respectively, and one may refer to [23, 18, 19, 26, 2, 13, 12] for more details. Here we omit the proofs.

Lemma 2.12. (9) For any \( r \geq 1 \) and for arbitrary \( L^{2}(P, H) \)-valued process \( \phi(\cdot) \) such that

\[
\sup_{s \in [0, t]} E \left\| \int_{0}^{s} \phi(u)dw(u) \right\|^{2r} \leq C_{r} \left( \int_{0}^{t} (E|\phi(s)|^{2r})^{\frac{1}{r}} ds \right)^{r}, \quad t \geq 0,
\]

where \( C_{r} = (r(2r - 1))^{r} \).

3 Almost periodic solutions for linear impulsive stochastic differential equations

To begin, consider the following linear impulsive stochastic differential equation:

\[
\begin{align*}
\frac{dx(t)}{dt} &= [Ax(t) + f(t)]dt + g(t)dw(t), \quad t \geq 0, t \neq \tau_{i}, i \in Z^{+}, \\
\Delta x(t_{i}) &= x(t_{i}^{+}) - x(t_{i}^{-}) = \beta_{i}, i \in Z^{+},
\end{align*}
\]
where A is an infinitesimal generator which generates a $C_0$-semigroup $\{T(t) : t \geq 0\}$ such that for all $t \geq 0$, $\|T(t)\| \leq Me^{-\delta t}$ with $M, \delta > 0$ and $\{T(t) : t > 0\}$ is compact. Furthermore, $f, g : \mathbb{R} \rightarrow L^2(P, H)$ are two stochastic processes, $\beta_i$ is a square-mean almost periodic sequence and $w(t)$ is a two-sided standard one-dimensional Brownian motion, which is defined on the filtered probability space $(\Omega, F, P, F_t)$ with $F_t = \sigma(\xi(u) - w(v) : u, v \leq t)$.

**Definition 3.1.** An $F_t$-progressive process $x(t)$ is called a mild solution of system (4) if it satisfies the following stochastic integral equation

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds + \int_0^t T(t-s)g(s)dw(s) + \sum_{0 < t_i < t} T(t-t_i)\beta_i,$$

for all $t \geq 0$.

**Theorem 3.2.** Assume $f, g \in AP(T, L^2(P, H))$, $\{\beta_i, i \in \mathbb{Z}^+\}$ is a square-mean almost periodic sequence, then system (4) has a square-mean piecewise almost periodic mild solution.

**Proof.** From semigroup theory, we know

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds + \int_0^t T(t-s)g(s)dw(s), t \geq 0,$$

is a mild solution to

$$dx(t) = [Ax(t) + f(t)]dt + g(t)dw(t), t \geq 0.$$

So for system (4), if $t \in [t_0, t_1)$,

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds + \int_0^t T(t-s)g(s)dw(s),$$

which implies

$$x(t^+) = T(t_1)x_0 + \int_0^{t_1} T(t_1-s)f(s)ds + \int_0^{t_1} T(t_1-s)g(s)dw(s),$$

by using $x(t^+) = x(t^-) + \beta_i$, for $t \in [t_1, t_2)$, we get

$$x(t) = T(t-t_1)x(t_1^+) + \int_{t_1}^{t} T(t-s)f(s)ds + \int_{t_1}^{t} T(t-s)g(s)dw(s)$$

$$= T(t-t_1)[x(t^-_1) + \beta_1] + \int_{t_1}^{t} T(t-s)f(s)ds + \int_{t_1}^{t} T(t-s)g(s)dw(s)$$

$$= T(t-t_1)[T(t_1)x_0 + \int_0^{t_1} T(t_1-s)f(s)ds + \int_0^{t_1} T(t_1-s)g(s)dw(s) + \beta_1]$$

$$+ \int_{t_1}^{t} T(t-s)f(s)ds + \int_{t_1}^{t} T(t-s)g(s)dw(s)$$

$$= T(t)x_0 + \int_0^{t} T(t-s)f(s)ds + \int_0^{t} T(t-s)g(s)dw(s) + T(t-t_1)\beta_1.$$
reiterating this procedure, we can prove that

\[ x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds + \int_0^t T(t-s)g(s)dw(s) + \sum_{0<t_i<t} T(t-t_i)\beta_i, \]

and by Definition \ref{def:mild}, \eqref{eq:5} is a mild solution of system \eqref{eq:4}, to finish the proof, we need to prove the above process \eqref{eq:5} is a square-mean piecewise almost periodic process.

Since \(f, g \in \text{AP}_T(\mathbb{R}^+, L^2(P, H))\), from Lemma 31 in \cite{Ref1}, for the two almost periodic functions \(f, g\), there exists a relatively dense set of their common \(\varepsilon\)-translation number. Moreover, \(\{\beta_i, t \in \mathbb{Z}^+\}\) is a square-mean almost periodic sequence, then by Lemma \ref{lem:2.11} for each \(\varepsilon > 0\), there exist relatively dense sets \(\Omega_{e, f, g, x_i} \subseteq \mathbb{R}\) and \(Q_{e, f, g, x_i} \subseteq \mathbb{Z}^+\) such that the following relations hold:

1. \(E\|f(t + \tau) - f(t)\|^2 < \varepsilon, t \in \mathbb{R}^+, |t - t_i| > \varepsilon, i \in \mathbb{Z}^+, \tau \in \Omega_{e, f, g, x_i}\).
2. \(E\|g(t + \tau) - g(t)\|^2 < \varepsilon, t \in \mathbb{R}^+, |t - t_i| > \varepsilon, i \in \mathbb{Z}^+, \tau \in \Omega_{e, f, g, x_i}\).
3. \(E\|x_{i+q} - x_i\|^2 < \varepsilon, i \in \mathbb{Z}^+, q \in Q_{e, f, g, x_i}\).
4. For each \(\tau \in \Omega_{e, f, g, x_i}\), \(\exists q \in Q_{e, f, g, x_i}\), s.t. \(|t_{i+q} - t_i - \tau| < \varepsilon, i \in \mathbb{Z}^+\).

We write \(x(t)\) of \eqref{eq:5} as

\[ x(t) = T(t)x_0 + x_1(t) + x_2(t) + x_3(t) \]

where

\[ x_1(t) = \int_0^t T(t-s)f(s)ds, \quad x_2(t) = \int_0^t T(t-s)g(s)dw(s), \quad x_3(t) = \sum_{0<t_i<t} T(t-t_i)\beta_i. \]

(i) \(x_1 \in \text{AP}_T(\mathbb{R}^+, L^2(P, H))\). By (1), for \(\tau \in \Omega_{e, f, g, x_i}, t \in \mathbb{R}^+, |t - t_i| > \varepsilon, i \in \mathbb{Z}^+\), one obtains

\[
E\|x_1(t + \tau) - x_1(t)\|^2 = E\left\|\int_0^t T(t-s)[f(s + \tau) - f(s)]ds\right\|^2 \\
\leq E\left[\int_0^t M e^{-\delta(t-s)}\|f(s + \tau) - f(s)\|ds\right]^2 \\
\leq E\left[\int_0^t M^2 e^{-\delta(t-s)}\int_0^t e^{-\delta(t-s)}\|f(s + \tau) - f(s)\|^2ds\right] \\
\leq \frac{M^2}{\delta}\int_0^t e^{-\delta(t-s)}E\|f(s + \tau) - f(s)\|^2ds \\
\leq \frac{M^2}{\delta}\int_0^t e^{-\delta(t-s)}\varepsilon ds \leq \frac{M^2}{\delta^2}\varepsilon.
\]

(ii) \(x_2 \in \text{AP}_T(\mathbb{R}^+, L^2(P, H))\). Let \(\tilde{w}(s) = w(s + \tau) - w(\tau)\) for each \(s \in \mathbb{R}^+\). Note that \(\tilde{w}\) is also...
a Brownian motion and has the same distribution as \( w \). By Lemma 2.12 and (2), we have

\[
E[|x_2(t + \tau) - x_2(t)|^2] = E\left(\int_0^t T(t - s)g(s + \tau)dw(s) - \int_0^t T(t - s)g(s)dw(s)\right)^2
\]

\[
= E\left(\int_0^t T(t - s)[g(s + \tau) - g(s)]d\tilde{w}(s)\right)^2
\]

\[
\leq \int_0^t E[|T(t - s)[g(s + \tau) - g(s)]|^2]ds
\]

\[
\leq \int_0^t M^2e^{-2\delta(t-s)}E[|g(s + \tau) - g(s)|^2]ds
\]

\[
\leq \int_0^t M^2e^{-2\delta(t-s)}\epsilon ds = \frac{M^2}{2\delta}\epsilon.
\]

(iii) \( x_3 \in AP_\Gamma(R^+, L^2(P, H)) \). Define

\[
r(t) = T(t - t_i)\beta_i, \quad t_i < t \leq t_{i+1}, i \in Z^+.
\]

For \( t_i < t \leq t_{i+1}, |t - t_i| > \epsilon, |t - t_{i+1}| > \epsilon, i \in Z^+ \), by (4), we can get

\[
t + \tau > t_i + \epsilon + \tau > t_{i+1},
\]

and

\[
t_{i+q+1} > t_{i+1} + \tau - \epsilon > t + \tau,
\]

that is, \( t_{i+q+1} > t + \tau > t_{i+1} \). Since \( (a + b)^2 \leq 2(a^2 + b^2) \), one has

\[
E[r(t + \tau) - r(t)]^2
\]

\[
= E[|T(t + \tau - t_{i+q})\beta_i + q - T(t - t_i)\beta_i|^2]
\]

\[
= E[|T(t + \tau - t_{i+q}) - T(t - t_i)|\beta_i|^2 + T(t - t_i)|\beta_i|^2 - 2T(t - t_i)|\beta_i|^2]
\]

\[
\leq 2E[|T(t + \tau - t_{i+q}) - T(t - t_i)|^2E[|\beta_i|^2] + 2|T(t - t_i)|E[|\beta_i|^2 - \beta_i|^2]
\]

\[
\leq 2|T(t + \tau - t_{i+q}) - T(t - t_i)|^2E[|\beta_i|^2] + 2M^2\epsilon,
\]

since \( \{T(t) : t \geq 0\} \) is a C_0-semigroup (see [21]), for the above \( \epsilon \), there exists \( 0 < \mu < \epsilon < 1 \) such that \( 0 < s < \mu \) implies \( \|T(t - t_i + s) - T(t - t_i)\| < \epsilon \). Note that \( M_0 = \sup_{i \in Z} E[|\beta_i|^2] < \infty \), so

\[
E[r(t + \tau) - r(t)]^2 \leq 2M_0\epsilon^2 + 2M^2\epsilon.
\]

Next we will prove that \( r \) is uniformly continuous on each interval \((t_i, t_{i+1})(i \in Z^+)\). Let \( t, h \in R^+ \) such that \( t_i < t, t + h < t_{i+1}, \) then

\[
E[r(t + h) - r(t)]^2 \leq \|T(t + h - t_i) - T(t - t_i)\|^2E[|\beta_i|^2].
\]
Since \( \{T(t) : t \geq 0 \} \) is a \( C_0 \)-semigroup and \( M_0 = \sup_{t \in \mathbb{Z}^+} E\|\beta_i\|^2 < \infty \), we conclude that \( E\|r(t + h) - r(t)\|^2 \to 0 \) as \( h \to 0 \) independent of \( t \) and \( i \).

Finally, by Cauchy-Schwarz inequality and (3),

\[
E \left| \sum_{0 < t_i < t + \tau} T(t + \tau - t_i)\beta_i - \sum_{0 < t_i < t} T(t - t_i)\beta_i \right|^2 \\
\leq E \left[ \sum_{0 < t_i < t} ||T(t - t_i)(\beta_{i + q} - \beta_i)||^2 \right] \\
\leq E \left[ \sum_{0 < t_i < t} M e^{-\delta(t-t_i)}||\beta_{i + q} - \beta_i||^2 \right] \\
\leq E \left[ \left( \sum_{0 < t_i < t} M^2 e^{-\delta(t-t_i)} \right) \sum_{0 < t_i < t} e^{-\delta(t-t_i)}||\beta_{i + q} - \beta_i||^2 \right] \\
\leq \frac{M^2}{1 - e^{-\delta\gamma}} \sum_{0 < t_i < t} e^{-\delta(t-t_i)} E||\beta_{i + q} - \beta_i||^2 \\
\leq \frac{M^2}{1 - e^{-\delta\gamma}} \sum_{0 < t_i < t} e^{-\delta(t-t_i)} \leq \frac{M^2}{(1 - e^{-\delta\gamma})^2} \varepsilon.
\]

In view of the above, it is clear that \( x_3 \in \text{AP}_T(\mathbb{R}^+,L^2(P,H)) \).

Furthermore, since \( \{T(t) : t \geq 0 \} \) is a bounded \( C_0 \)-semigroup and \( \{T(t) : t > 0 \} \) is compact, by Theorem 2.1 in [5], \( T(\cdot)x_0 \in \text{AP}_T(\mathbb{R}^+,L^2(P,H)) \). By combing (i), (ii) and (iii), it follows that \( x_3 \) is a square-mean piecewise almost periodic process, so system (4) has a square-mean piecewise almost periodic solution. The proof is complete. \( \square \)

4 Almost periodic solutions for nonlinear impulsive stochastic differential equations

Consider the following nonlinear impulsive stochastic differential equation

\[
\begin{aligned}
\text{dx}(t) &= [Ax(t) + f(t,x(t))]dt + g(t,x(t))dw(t), & t \geq 0, t \neq t_i, i \in \mathbb{Z}^+, \\
\Delta x(t_i) &= x(t_i^+) - x(t_i^-) = I_i(x(t_i)), & i \in \mathbb{Z}^+,
\end{aligned}
\]

(6)

where \( f, g : \mathbb{R}^+ \times L^2(P,H) \to L^2(P,H), \ I_i : L^2(P,H) \to L^2(P,H), i \in \mathbb{Z}^+ \) and \( w(t) \) is a two-sided standard one dimensional Brownian motion defined on the filtered probability space \( (\Omega,F,F,F_\sigma) \) with \( F_t = \sigma(w(u) - w(v) : u,v \leq t) \).

Definition 4.1. An \( F_t \)-progressive process \( x(t) \) is called a mild solution of system (6) if it satisfies
the corresponding stochastic integral equation

\[ x(t) = T(t)x_0 + \int_0^t T(t-s)f(s, x(s))ds + \int_0^t T(t-s)g(s, x(s))dw(s) + \sum_{0 < t_i < t} T(t-t_i)I_i(x(t_i)). \] (7)

for all \( t \geq 0 \).

In order to obtain the existence of square-mean piecewise almost periodic solution to system \( \text{(6)} \), we introduce the following assumptions:

(A1) The operator \( A : D(A) \subseteq L^2(P, H) \to L^2(P, H) \) is the infinitesimal generator of an exponentially stable \( C_0 \)-semigroup \( \{T(t) : t \geq 0 \} \) on \( L^2(P, H) \), i.e., \( \|T(t)\| \leq Me^{-\delta t}, \ t \geq 0, \ M, \delta > 0 \). Moreover, \( T(t) \) is compact for \( t > 0 \).

(A2) \( f, g \in AP_T([R^+ \times L^2(P, H), L^2(P, H)]) \), for each compact set \( K \subseteq L^2(P, H) \), \( g(t, \cdot), f(t, \cdot) \) are uniformly continuous in each compact set \( K \subseteq L^2(P, H) \) uniformly for \( t \in R^+ \). \( I_i(x) \) is almost periodic in \( i \in Z^+ \) uniformly in \( x \in K \) and is a uniformly continuous function defined on the set \( K \subseteq L^2(P, H) \) for all \( i \in Z^+ \).

(A3) \( F_L = \sup_{t \in R^+, E||x|| \leq L} E||f(t, x)||^2 < \infty, \ G_L = \sup_{t \in R^+, E||x|| \leq L} E||g(t, x)||^2 < \infty, \ I_L = \sup_{t \in Z^+, E||x|| \leq L} E||I_i(x(t))||^2 < \infty \), where \( L \) is an arbitrary positive number. Moreover, there exist a number \( L_0 > 0 \) such that \( 4M^2L_0 + \frac{4M^2}{\delta}F_{L_0} + \frac{2M^2}{\delta}G_{L_0} + \frac{4M^2}{(1 - e^{-\delta L})}I_{L_0} \leq L_0 \).

**Theorem 4.2.** Assume that the conditions (A1)-(A3) are satisfied, then the impulsive stochastic differential equation \( \text{(6)} \) admits at least one square-mean piecewise almost periodic solution.

**Proof.** Let

\[ B = \{ x \in AP_T([R^+, L^2(P, H)]) : E||x||^2 \leq L_0 \}. \]

Obviously, \( B \) is a closed set of \( AP_T([R^+, L^2(P, H)]) \). Define \( \Gamma \) on \( (PC)_h^0([R^+, L^2(P, H)]) \),

\[ \Gamma x(t) = T(t)x_0 + \int_0^t T(t-s)f(s, x(s))ds + \int_0^t T(t-s)g(s, x(s))dw(s) + \sum_{0 < t_i < t} T(t-t_i)I_i(x(t_i)). \]

In order to show that the impulsive stochastic differential equation \( \text{(6)} \) has a square-mean piecewise almost periodic solution, we only need to prove the operator \( \Gamma \) has a fixed point in \( B \).

First we show \( \Gamma x \in B, x \in B \). For \( x \in B \), by Lemma \( \text{(2,6)} \) and (A2), we have \( f(\cdot, x(\cdot)), g(\cdot, x(\cdot)) \in AP_T([R^+, L^2(P, H)]) \), by (A2) and Lemma 37 in \( \text{[23]} \), \( I_i(x(t_i)) \) is a square-mean almost periodic sequence, analogous to the proof of Theorem \( \text{[4,2]} \) we can show \( \Gamma x \in AP_T([R^+, L^2(P, H)]) \).

Since \( (a+b+c+d)^2 \leq 4(a^2 + b^2 + c^2 + d^2) \), by using Cauchy-Schwarz inequality and Lemma
2.12 We obtain
\[ E\|f(x(t))\|^2 \leq 4E\|T(t)x_0\|^2 + 4E\left\| \int_0^t T(t-s)f(s,x(s))ds \right\|^2 + 4E\left\| \int_0^t T(t-s)g(s,x(s))dw(s) \right\|^2 \]
\[ + 4E\left\| \sum_{0<t_1<t} T(t-t_1)I_1(x(t_1)) \right\|^2 \]
\[ \leq 4M^2E\|x_0\|^2 + 4E\left[ \int_0^t M^2e^{-\delta(t-s)}\|f(s,x(s))\|ds \right]^2 + 4\int_0^t E\|T(t-s)g(s,x(s))\|^2ds \]
\[ + 4\sum_{0<t_1<t} M^2e^{-\delta(t-t_1)}\|I_1(x(t_1))\|^2 \]
\[ \leq 4M^2L_0 + 4E\left[ \int_0^t M^2e^{-\delta(t-s)}ds \int_0^t e^{-\delta(t-s)}\|f(s,x(s))\|^2ds \right] + 4\int_0^t M^2e^{-2\delta(t-s)}E\|g(s,x(s))\|^2ds \]
\[ + 4\sum_{0<t_1<t} M^2e^{-\delta(t-t_1)}\|I_1(x(t_1))\|^2 \]
\[ \leq 4M^2L_0 + 4M^2\delta \int_0^t e^{-\delta(t-s)}F_{L_0}ds + 4\int_0^t M^2e^{-2\delta(t-s)}G_{L_0}ds + \frac{4M^2}{1-e^{-2\delta}} \sum_{0<t_1<t} e^{-\delta(t-t_1)}I_{L_0} \]
\[ \leq 4M^2L_0 + \frac{4M^2}{\delta^2}F_{L_0} + \frac{2M^2}{\delta}G_{L_0} + \frac{4M^2}{1-e^{-2\delta}}I_{L_0}, \]

since \(4M^2L_0 + \frac{4M^2}{\delta^2}F_{L_0} + \frac{2M^2}{\delta}G_{L_0} + \frac{4M^2}{1-e^{-2\delta}}I_{L_0} \leq L_0\), then \(E\|f\|^2 \leq L_0\), that is, \(f \in L^2(P,H)\) for each \(t \in \mathbb{R}^+\).

Next we show that \(B(t) = \{f(t) : x \in B \}\) is a relatively compact subset of \(L^2(P,H)\) for each \(t \in \mathbb{R}^+\).

For each \(t \in \mathbb{R}^+\), \(0 < \epsilon < 1\), \(x \in B\), define
\[ \Gamma^\epsilon x(t) = T(t)x_0 + \int_0^{t-\epsilon} T(t-s)f(s,x(s))ds + \int_0^{t-\epsilon} T(t-s)g(s,x(s))dw(s) \]
\[ + \sum_{0<t_1<t-\epsilon} T(t-t_1)I_1(x(t_1)) \]
\[ = T(\epsilon)T(t-\epsilon)x_0 + \int_0^{t-\epsilon} T(t-\epsilon-s)f(s,x(s))ds \]
\[ + \int_0^{t-\epsilon} T(t-\epsilon-s)g(s,x(s))dw(s) + \sum_{0<t_1<t-\epsilon} T(t-\epsilon-t_1)I_1(x(t_1)) \]
\[ = T(\epsilon)\Gamma^\epsilon x(t-\epsilon). \]

Since \(f(x(t-\epsilon)) : x \in B\) is bounded and \(T(\epsilon)\) is compact, \(\{\Gamma^\epsilon x(t) : x \in B\}\) is a relatively compact subset of \(L^2(P,H)\). Moreover, for \(\epsilon\) is small enough and the points \(t\) and \(t-\epsilon\) belong to the same
interval of continuity of $x$, then

\[ \Gamma(x(t) - \Gamma^e x(t)) = \int_{t - \varepsilon}^t T(t - s)f(s, x(s))ds + \int_{t - \varepsilon}^t T(t - s)g(s, x(s))dw(s), \]

since $(a + b)^2 \leq 2(a^2 + b^2)$, by using Cauchy-Schwarz inequality and Lemma 2.12, one has

\[
\begin{align*}
\mathbb{E}[\|\Gamma x(t) - \Gamma^e x(t)\|^2] & \leq 2 \left[ \mathbb{E}\left[ \left\| \int_{t - \varepsilon}^t T(t - s)f(s, x(s))ds \right\|^2 \right] + \mathbb{E}\left[ \left\| \int_{t - \varepsilon}^t T(t - s)g(s, x(s))dw(s) \right\|^2 \right] \right] \\
& \leq 2E \left[ \int_{t - \varepsilon}^t M e^{-\delta(t-s)}\|f(s, x(s))\|ds \right]^2 + 2 \int_{t - \varepsilon}^t E\|T(t - s)g(s, x(s))\|^2ds \\
& \leq 2E \left[ \int_{t - \varepsilon}^t M^2 e^{-\delta(t-s)}ds \right] \int_{t - \varepsilon}^t e^{-\delta(t-s)}\|f(s, x(s))\|^2ds + 2 \int_{t - \varepsilon}^t M^2 e^{-\delta(t-s)}E\|g(s, x(s))\|^2ds \\
& \leq 2M^2\varepsilon \int_{t - \varepsilon}^t \mathbb{E}\|f(s, x(s))\|^2ds + 2M^2 \int_{t - \varepsilon}^t \mathbb{E}\|g(s, x(s))\|^2ds \\
& \leq 2M^2\varepsilon^2F_{\varepsilon} + 2M^2\varepsilon G_{\varepsilon},
\end{align*}
\]

so $B(t) = \{\Gamma x(t) : x \in B\}$ is a relatively compact subset of $L^2(P, \mathcal{H})$ for each $t \in \mathbb{R}^+$. Finally we show $\{\Gamma x : x \in B\}$ is equicontinuous at each interval $(t_i, t_{i+1}) (i \in \mathbb{Z}^+)$. Let $x \in B$, $t_i < t'' < t' < t_{i+1}, i \in \mathbb{Z}^+$, and $\rho < \min \left\{ \frac{\varepsilon}{36M^2F_{\varepsilon}}, \frac{\varepsilon}{36M^2G_{\varepsilon}}, 1 \right\}$.

\[
\begin{align*}
\Gamma x(t') - \Gamma x(t'') &= T(t')x_0 + \int_0^{t'} T(t' - s)f(s, x(s))ds + \int_0^{t'} T(t' - s)g(s, x(s))dw(s) \\
&\quad + \sum_{0 < t_i < t'} T(t' - t_i)I_i(x(t_i)) - T(t'')x_0 - \int_0^{t''} T(t'' - s)f(s, x(s))ds \\
&\quad - \int_0^{t''} T(t'' - s)g(s, x(s))dw(s) - \sum_{0 < t_i < t''} T(t'' - t_i)I_i(x(t_i)) \\
&= T(t') - T(t'')x_0 + \int_{t''}^{t'} T(t' - s)f(s, x(s))ds + \int_{t''}^{t'} T(t' - s)g(s, x(s))dw(s) \\
&\quad + \int_0^{t''} [T(t' - s) - T(t'' - s)]f(s, x(s))ds + \int_0^{t''} [T(t' - s) - T(t'' - s)]g(s, x(s))dw(s) \\
&\quad + \sum_{0 < t_i < t''} [T(t' - t_i) - T(t'' - t_i)]I_i(x(t_i)).
\end{align*}
\]

Since $\{T(t) : t \geq 0\}$ is a $C_0$-semigroup, there exists $\mu < \rho$ such that $t' - t'' < \mu$ implies that

\[
\begin{align*}
\|T(t) - I\|^2 &\leq \min \left\{ \frac{\varepsilon}{36M^2L_0}, \frac{\varepsilon\delta^2}{36M^2F_{\varepsilon}}, \frac{\varepsilon\delta}{18M^2G_{\varepsilon}}, \frac{\varepsilon(1 - e^{-\delta\mu})^2}{36M^2L_{\varepsilon}} \right\}.
\end{align*}
\]
By using Cauchy-Schwarz inequality and Lemma 2.12, we have

\[
E[|T(t') - T(t'')]|x_0|^2 \leq E[|T(t' - t'') - I|T(t'')x_0|^2
\]

\[
\leq |T(t' - t'') - I|^2 M^2 E|x_0|^2
\]

\[
\leq \frac{\epsilon}{36M^2L_0} M^2 L_0 = \frac{\epsilon}{36}
\]

and,

\[
E \left[ \left\| \int_{t''}^{t'} T(t' - s)f(s, x(s))ds \right\|^2 \right] \leq E \left[ \int_{t''}^{t'} M^2 e^{-\delta(t' - s)} \left\| f(s, x(s)) \right\|^2 ds \right]
\]

\[
\leq M^2 (t' - t'') \int_{t''}^{t'} e^{-\delta(t' - s)} \left\| f(s, x(s)) \right\|^2 ds
\]

\[
\leq M^2 F_{L_0} (t' - t'')^2
\]

\[
\leq M^2 F_{L_0} \frac{\epsilon}{36M^2L_0} = \frac{\epsilon}{36},
\]

and

\[
E \left[ \left\| \int_{t''}^{t'} T(t' - s)g(s, x(s))dw(s) \right\|^2 \right] \leq \int_{t''}^{t'} E[|T(t' - s)g(s, x(s))|^2] ds
\]

\[
\leq \int_{t''}^{t'} M^2 e^{-2\delta(t' - s)} E[|g(s, x(s))|^2] ds
\]

\[
\leq M^2 G_{L_0} (t' - t'')
\]

\[
\leq M^2 G_{L_0} \frac{\epsilon}{36M^2G_{L_0}} = \frac{\epsilon}{36},
\]

and

\[
E \left[ \left\| \int_{0}^{t''} [T(t' - s) - T(t'') - s]f(s, x(s))ds \right\|^2 \right]
\]

\[
= E \left[ \left\| \int_{0}^{t''} [T(t' - s) - I]T(t'')f(s, x(s))ds \right\|^2 \right]
\]

\[
\leq E \left[ \int_{0}^{t''} \left\| T(t' - s) - I \right\| M^2 e^{-\delta(t' - s)} \left\| f(s, x(s)) \right\|^2 ds \right]^2
\]

\[
\leq E \left[ \int_{0}^{t''} \left\| T(t' - s) - I \right\|^2 M^2 e^{-\delta(t' - s)} \left\| f(s, x(s)) \right\|^2 ds \right]
\]

\[
\leq |T(t' - t'') - I|^2 \frac{M^2}{\delta} \int_{0}^{t''} e^{-\delta(t' - s)} E[|f(s, x(s))|^2] ds
\]

\[
\leq |T(t' - t'') - I|^2 \frac{M^2}{\delta} \int_{0}^{t''} e^{-\delta(t' - s)} F_{L_0} ds
\]

\[
\leq \frac{\epsilon \delta^2}{36M^2F_{L_0}} \frac{M^2}{\delta} F_{L_0} = \frac{\epsilon}{36}.
\]
and

\[
E \left\| \int_0^{t''} [T(t' - s) - T(t'' - s)] g(s, x(s)) \, dw(s) \right\|^2 \\
\leq \int_0^{t''} E \left\| [T(t' - t'') - I] T(t'' - s) g(s, x(s)) \right\|^2 \, ds \\
\leq \int_0^{t''} \|T(t' - t'') - I\|^2 M^2 e^{-2\delta(t''-s)} E||f(s, x(s))||^2 \, ds \\
\leq \|T(t' - t'') - I\|^2 \int_0^{t''} M^2 e^{-2\delta(t''-s)} G_{L_0} \, ds \\
\leq \frac{e\delta}{18M^2G_{L_0}} \frac{M^2}{2\delta} G_{L_0} = \frac{e}{36},
\]

and

\[
E \left\| \sum_{0 < t_1 < t''} [T(t' - t_1) - T(t'' - t_1)] I_1(x(t_1)) \right\|^2 \\
= E \left\| \sum_{0 < t_1 < t''} [T(t' - t'') - I] T(t'' - t_1) I_1(x(t_1)) \right\|^2 \\
\leq E \left[ \sum_{0 < t_1 < t''} \|T(t' - t'') - I\|^2 \|T(t'' - t_1)\|^2 \|I_1(x(t_1))\|^2 \right] \\
\leq E \left[ \left( \sum_{0 < t_1 < t''} \|T(t' - t'') - I\|^2 e^{-\delta(t''-t_1)} \right) \sum_{0 < t_1 < t''} e^{-\delta(t''-t_1)} \|I_1(x(t_1))\|^2 \right] \\
\leq \|T(t' - t'') - I\|^2 \frac{M^2}{1 - e^{-\delta\gamma}} \sum_{0 < t_1 < t''} e^{-\delta(t''-t_1)} \|I_1(x(t_1))\|^2 \\
\leq \|T(t' - t'') - I\|^2 \frac{M^2}{1 - e^{-\delta\gamma}} \sum_{0 < t_1 < t''} e^{-\delta(t''-t_1)} I_{L_0} \\
\leq \frac{e(1 - e^{-\delta\gamma})^2}{36M^2I_{L_0}} \frac{M^2}{(1 - e^{-\delta\gamma})^2} I_{L_0} = \frac{e}{36},
\]
so that, for \( x \in B \) and \( t' - t'' < \mu, t', t'' \in (t_i, t_{i+1}), i \in \mathbb{Z} \),

\[
\mathbb{E}||f(t') - f(t'')||^2 \\
\leq 6\mathbb{E}||T(t') - T(t'')x_0||^2 + 6\mathbb{E}\left|\int_{t'}^{t''} T(t' - s)f(s, x(s))ds\right|^2 \\
+ 6\mathbb{E}\left|\int_{t'}^{t''} T(t' - s)g(s, x(s))dw(s)\right|^2 + 6\mathbb{E}\left|\int_0^{t''} [T(t' - s) - T(t'' - s)]f(s, x(s))ds\right|^2 \\
+ 6\mathbb{E}\left|\sum_{0 < t_i < t''} [T(t' - t_i) - T(t'' - t_i)]I_i(x(t_i))\right|^2 \\
\leq \epsilon,
\]

which shows that \( \{fx : x \in B\} \) is equicontinuous at each interval \((t_i, t_{i+1})(i \in \mathbb{Z}^+)\).

Since \( \{fx : x \in B\} \subseteq \{PC_t^\mathbb{R}^+, L^2(P, H)\} \) and \( \{fx : x \in B\} \) is a relatively compact set, moreover, the Lebesgue dominated convergence theorem and our assumptions on \( f, g \) and \( I_i \) imply that \( \Gamma \) is continuous, then \( \Gamma \) is a compact operator. It follows from Schauder fixed point theorem that \( \Gamma \) has a fixed point in \( B \). Thus \( x \) is a square-mean piecewise almost periodic solution of system \( (6) \). The proof is complete. \( \square \)

Note that the uniformly continuous is weaker than the Lipschitz continuous, if \( (A2) \) is replaced by the following condition:

\( (A2') \) \( f, g \in AP_T^\mathbb{R}^+(P^+, L^2(P, H)), I_i(x) \) is almost periodic in \( i \in \mathbb{Z}^+ \) uniformly for \( x \in L^2(P, H) \), and there exists positive numbers \( L_1, L_2, L \) such that

\[
\mathbb{E}||f(t, x) - f(t, y)||^2 \leq L_1 \mathbb{E}||x - y||^2, \\
\mathbb{E}||g(t, x) - g(t, y)||^2 \leq L_2 \mathbb{E}||x - y||^2, \\
\mathbb{E}||I_i(x) - I_i(y)||^2 \leq L \mathbb{E}||x - y||^2,
\]

for all \( x, y \in L^2(P, H), t \in \mathbb{R}^+ \),

then by Lemma \( 2.7 \) and Theorem \( 3.2 \) we can also get the almost periodic solution of system \( (6) \).

**Corollary 4.3.** Suppose that the conditions \( (A1), (A2') \) and \( (A3) \) are satisfied, then the impulsive stochastic differential equation \( (6) \) has a square-mean piecewise almost periodic solution.

## 5 Stability

In this section we consider the stability of square-mean piecewise almost periodic solution to system \( (6) \) with Lipschitz activation function. In the sequel, we will need the following lemma.
Lemma 5.1. \((23)\) Let a nonnegative piecewise continuous function \(u(t)\) satisfy for \(t \geq t_0\) the inequality
\[
u(t) \leq C + \int_{t_0}^{t} v(\tau)u(\tau)\,d\tau + \sum_{t_0 < \tau_i < t} \beta_i u(\tau_i),
\]
where \(C \geq 0, \beta_i \geq 0, v(\tau) > 0,\) and \(\tau_i\)'s are discontinuity points of first type of the function \(u(t)\). Then the following estimate holds for the function \(u(t)\),
\[
u(t) \leq C \prod_{t_0 < \tau_i < t} (1 + \beta_i) e^{\int_{t_0}^{\tau_i} v(\tau)\,d\tau}.
\]

Theorem 5.2. Assume the conditions of Corollary \((4.3)\) are fulfilled. Assume further that \(\frac{1}{\gamma} \ln(1 + \frac{4M^2L_1}{\gamma - e^{-\delta \gamma}}) + \frac{4M^2L_1}{\delta} + 4M^2L_2 < 0\). Then system \((6)\) has an exponentially stable almost periodic solution.

Proof. By Corollary \((4.3)\) system \((6)\) has a mild square-mean piecewisely almost periodic solution \(u(t)\),
\[
u(t) = T(t)u_0 + \int_{0}^{t} T(t-s)f(s,u(s))\,ds + \int_{0}^{t} T(t-s)g(s,u(s))\,dw(s) + \sum_{0 < \tau_i < t} T(t-t_i)I_i(u(t_i)).
\]

Let \(u(t) = u(t,0,\varphi)\) and \(v(t) = v(t,0,\psi)\) be two solutions of equation \((6)\), then
\[
u(t) = T(t)\varphi + \int_{0}^{t} T(t-s)f(s,u(s))\,ds + \int_{0}^{t} T(t-s)g(s,u(s))\,dw(s) + \sum_{0 < \tau_i < t} T(t-t_i)I_i(u(t_i)),
\]
\[
\psi(t) = T(t)\psi + \int_{0}^{t} T(t-s)f(s,v(s))\,ds + \int_{0}^{t} T(t-s)g(s,v(s))\,dw(s) + \sum_{0 < \tau_i < t} T(t-t_i)I_i(v(t_i)).
\]

Since \((a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2)\), by \((A2')\), Cauchy-Schwarz inequality and Lemma
we have

\[
E|u(t) - v(t)|^2 \\
= E\left| T(t)\varphi - T(t)\psi + \int_0^t T(t-s)[f(s,u(s)) - f(s,v(s))]ds \\
+ \int_0^t T(t-s)[g(s,u(s)) - g(s,v(s))]dw(s) + \sum_{0 < t_i < t} T(t - t_i)[I_i(u(t_i)) - I_i(v(t_i))] \right|^2 \\
\leq 4E\|T(t)\varphi - \psi\|^2 + 4E\left| \int_0^t T(t-s)[f(s,u(s)) - f(s,v(s))]ds \right|^2 \\
+ 4E\left| \int_0^t T(t-s)[g(s,u(s)) - g(s,v(s))]dw(s) \right|^2 \\
+ 4E\left| \sum_{0 < t_i < t} T(t - t_i)[I_i(u(t_i)) - I_i(v(t_i))] \right|^2 \\
\leq 4M^2e^{-2\delta t}E|\varphi - \psi|^2 + 4E\left[ \int_0^t M^2 e^{-\delta(t-s)}ds \int_0^t e^{-\delta(t-s)}|f(s,u(s)) - f(s,v(s))|^2 ds \right] \\
+ 4\int_0^t M^2 e^{-2\delta(t-s)}E|g(s,u(s)) - g(s,v(s))|^2 ds \\
+ 4E\left[ \sum_{0 < t_i < t} M^2 e^{-\delta(t-t_i)} \sum_{0 < t_i < t} e^{-\delta(t-t_i)}|I_i(u(t_i)) - I_i(v(t_i))|^2 \right] \\
\leq 4M^2e^{-2\delta t}E|\varphi - \psi|^2 + \frac{4M^2}{\delta} \int_0^t e^{-\delta(t-s)}E|f(s,u(s)) - f(s,v(s))|^2 ds \\
+ 4M^2 \int_0^t e^{-\delta(t-s)}E|g(s,u(s)) - g(s,v(s))|^2 ds \\
+ \frac{4M^2}{1 - e^{-\delta y}} \sum_{0 < t_i < t} e^{-\delta(t-t_i)}E|I_i(u(t_i)) - I_i(v(t_i))|^2 \\
\leq 4M^2e^{-2\delta t}E|\varphi - \psi|^2 + \left( \frac{4M^2L_1}{\delta} + 4M^2L_2 \right) \int_0^t e^{-\delta(t-s)}E|u(s) - v(s)|^2 ds \\
+ \frac{4M^2L_1}{1 - e^{-\delta y}} \sum_{0 < t_i < t} e^{-\delta(t-t_i)}E|u(t_i) - v(t_i)|^2.
\]
Then,
\[
e^{\delta t} E \|u(t) - v(t)\|^2 \leq 4M^2 E \|\varphi - \psi\|^2 + \left( \frac{4M^2 L_1}{\delta} + 4M^2 L_2 \right) \int_0^t e^{\delta s} E \|u(s) - v(s)\|^2 ds
\]
\[
+ \frac{4M^2 L}{1 - e^{-\delta \gamma}} \sum_{0 < t_i < t} e^{\delta t_i} E \|u(t_i) - v(t_i)\|^2.
\]

Let \( \Upsilon(t) = e^{\delta t} E \|u(t) - v(t)\|^2 \), then
\[
\Upsilon(t) \leq 4M^2 \Upsilon(0) + \left( \frac{4M^2 L_1}{\delta} + 4M^2 L_2 \right) \int_0^t \Upsilon(s) ds + \frac{4M^2 L}{1 - e^{-\delta \gamma}} \sum_{0 < t_i < t} \Upsilon(t_i).
\]

By Lemma 5.1 we have
\[
\Upsilon(t) \leq 4M^2 \Upsilon(0) \prod_{0 < t_i < t} \left( 1 + \frac{4M^2 L}{1 - e^{-\delta \gamma}} \right) e^{\int_0^t \left( \frac{4M^2 L_1}{\delta} + 4M^2 L_2 \right) ds}
\]
\[
= 4M^2 \Upsilon(0) \prod_{0 < t_i < t} \left( 1 + \frac{4M^2 L}{1 - e^{-\delta \gamma}} \right) e^{\left( \frac{4M^2 L_1}{\delta} + 4M^2 L_2 \right) t}
\]
\[
\leq 4M^2 \Upsilon(0) (1 + \frac{4M^2 L}{1 - e^{-\delta \gamma}}) e^{\left( \frac{1}{\gamma} \ln(1 + \frac{4M^2 L_1}{1 - e^{-\delta \gamma}}) + \frac{4M^2 L_1}{\delta} + 4M^2 L_2 \right) t},
\]

that is,
\[
E \|u(t) - v(t)\|^2 \leq 4M^2 E \|\varphi - \psi\|^2 e^{\left( \frac{1}{\gamma} \ln(1 + \frac{4M^2 L_1}{1 - e^{-\delta \gamma}}) + \frac{4M^2 L_1}{\delta} + 4M^2 L_2 \right) t}.
\]

Since \( \frac{1}{\gamma} \ln(1 + \frac{4M^2 L_1}{1 - e^{-\delta \gamma}}) + \frac{4M^2 L_1}{\delta} + 4M^2 L_2 < 0 \). The square-mean piecewise almost periodic solution of system (6) is exponentially stable. This completes the proof.

\[\square\]

References


Existence and stability of almost periodic solutions to impulsive ...


Approximation by Discrete Singular Operators

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ABSTRACT

Here we study basic approximation properties with rates of our discrete versions of Picard, Gauss-Weierstrass, Poisson-Cauchy singular operators and of two other discrete operators. We prove uniform convergence of these operators to the unit operator. Also all these operators fulfill the global smoothness preservation property. The discussed operators act on the space of uniformly continuous functions over the real line.

RESUMEN

Aquí estudiamos las propiedades de aproximación básica con cocientes de nuestras versiones discretas de operadores singulares de Picard, Gauss-Weierstrass, Poisson-Cauchy y de otros dos operadores discretos. Probamos la convergencia uniforme de estos operadores al operador unitario. Además, todos estos operadores satisfacen la propiedad de preservación de suavidad global. Los operadores discutidos actúan sobre el espacio de funciones uniformemente continua sobre la recta real.

Keywords and Phrases: Discrete singular operator, modulus of continuity, uniform convergence, global smoothness.

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1 Preliminaries

Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is uniformly continuous ($f \in C_{u}(\mathbb{R})$). Following [2], p. 40-41, we define the first modulus of continuity,

$$
\omega_{1}(f, t) := \sup_{x, y \in \mathbb{R}} |f(x) - f(y)|, \quad t \geq 0.
$$

(1)

The function $\omega_{1}$ is continuous at $t = 0$ if and only if $f$ is uniformly continuous on $\mathbb{R}$. So that here $\omega_{1}(f, t) \to \omega_{1}(f, 0) = 0$, as $t \to 0$. It also holds

$$
\omega_{1}(f, \lambda t) \leq (\lambda + 1) \omega_{1}(f, t), \quad \lambda \geq 0.
$$

(2)

Clearly $\omega_{1}(f, t)$ is finite for each $t \geq 0$.

In [1] we studied extensively the convergence to the unit operator of various integral singular operators. Here we define the discrete analogs of these operators next, and we study their uniform convergence to the unit operator with rates.

Let $0 < \xi, \leq 1$, such that $\xi \to 0+, \ x \in \mathbb{R}; \ \frac{1}{\xi} \geq 1$.

i) We define the discrete Picard operators:

$$
(P_{\xi} f) (x) := \frac{\sum_{\nu=-\infty}^{\infty} f(x + \nu) e^{-|\nu| \xi}}{\sum_{\nu=-\infty}^{\infty} e^{-|\nu| \xi}}.
$$

(3)

ii) We define the discrete Gauss-Weierstrass operators:

$$
(W_{\xi} f) (x) := \frac{\sum_{\nu=-\infty}^{\infty} f(x + \nu) e^{-\frac{\nu^{2}}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^{2}}{\xi}}}.
$$

(4)

iii) We define the general discrete Poisson-Cauchy operators:

let $\alpha \in \mathbb{N}, \beta > 1$;

$$
(M_{\xi} f) (x) := \frac{\sum_{\nu=-\infty}^{\infty} f(x + \nu) \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}{\sum_{\nu=-\infty}^{\infty} \left(\nu^{2\alpha} + \xi^{2\alpha}\right)^{-\beta}}.
$$

(5)

iv) We define the basic discrete convolution operators:

let $\varphi : \mathbb{R} \to \mathbb{R}$, with $\|\varphi\|_{\infty} := \sup_{x \in \mathbb{R}} |\varphi(x)| \leq K$, $K > 0$, $\beta \in \mathbb{N} - \{1\}$;

$$
(\theta_{\xi} f) (x) := \frac{f(x) + \sum_{\nu \in \mathbb{Z} - \{0\}} f(x + \nu) \left(\frac{\varphi(\frac{\nu}{\xi})}{\xi}\right)^{2\beta} \frac{2}{1 + \sum_{\nu \in \mathbb{Z} - \{0\}} \left(\frac{\varphi(\frac{\nu}{\xi})}{\xi}\right)^{2\beta}}.
$$

(6)
v) We define the general discrete convolution operators:

let \( \varphi : \mathbb{R} \to \mathbb{R}_+ \) with \( \varphi (x) \leq Ax^{2\beta}, \forall x \in \mathbb{R}, \beta \in \mathbb{N} \setminus \{1\}, A > 0; \)

\[
(T_\xi^sf)(x) := \frac{f(x) + \sum_{\nu \in \mathbb{Z} \setminus \{0\}} f(x + \nu) \frac{\varphi(\frac{\nu}{\xi})}{(\frac{\nu}{\xi})^{4\beta}}}{1 + \sum_{\nu \in \mathbb{Z} \setminus \{0\}} \frac{\varphi(\frac{\nu}{\xi})}{(\frac{\nu}{\xi})^{4\beta}}}. \tag{7}
\]

The above operators, as we will see, are well defined and are linear, positive, and bounded when \( \|f\|_\infty := \sup_{x \in \mathbb{R}} |f(x)| < \infty. \) Furthermore

\[
P_\xi^s(1) = W_\xi^s(1) = M_\xi^s(1) = \theta_\xi^s(1) = T_\xi^s(1) = 1, \tag{8}
\]

with

\[
\|P_\xi^s\| = \|W_\xi^s\| = \|M_\xi^s\| = \|\theta_\xi^s\| = \|T_\xi^s\| = 1, \tag{9}
\]
on continuous bounded functions.

In this article we are motivated by [3].

### 2 Main Results

All here as in Preliminaries earlier. We start with the basic approximation properties of discrete Picard operators.

We present

**Theorem 2.1.** It holds

\[
\|P_\xi^sf - f\|_\infty \leq \left[ \frac{1 + 2e^{-\frac{1}{\xi}} \left(2\xi + 2 + \frac{1}{\xi}\right)}{1 + 2\xi e^{-\frac{1}{\xi}}} \right] \omega_1(f, \xi). \tag{10}
\]

The constant in the right hand side of (10) converges to 1 as \( \xi \to 0^+. \) So that \( P_\xi^s \to I \) (unit operator), uniformly with rates, as \( \xi \to 0^+. \)

**Proof.** We will use a lot

\[
\sum_{\nu=1}^{\infty} \frac{1}{\nu^2} = \frac{\pi^2}{6} \quad \text{(Euler, 1741).}
\]

We see that

\[
\sum_{\nu=-\infty}^{-1} e^{-|\nu|} = \sum_{\nu=1}^{\infty} e^{-\frac{\nu}{\xi}} < \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} = \frac{\pi^2}{6},
\]

it converges.
Thus
\[ \sum_{\nu=-\infty}^{\infty} e^{-\frac{|\nu|}{\xi}} = 2 \sum_{\nu=1}^{\infty} e^{-\frac{\nu}{\xi}} + 1 < \frac{\pi^2}{3} + 1. \]  
(11)

Using (4) we obtain
\[ \sum_{\nu=1}^{\infty} e^{-\frac{\nu}{\xi}} - e^{-\frac{1}{\xi}} \leq \int_{1}^{\infty} e^{-\frac{v}{\xi}} dv \leq \sum_{\nu=1}^{\infty} e^{-\frac{\nu}{\xi}}. \]  
(12)

Hence
\[ 2 \int_{1}^{\infty} e^{-\frac{v}{\xi}} dv + 1 \leq 2 \sum_{\nu=1}^{\infty} e^{-\frac{\nu}{\xi}} + 1 = \sum_{\nu=-\infty}^{\infty} e^{-\frac{|\nu|}{\xi}}. \]  
(13)

Thus
\[ 0 < \frac{1}{\sum_{\nu=-\infty}^{\infty} e^{-\frac{|\nu|}{\xi}}} \leq \frac{1}{2 \int_{1}^{\infty} e^{-\frac{v}{\xi}} dv + 1} = \frac{1}{2 \xi e^{-\frac{1}{\xi}} + 1} \to 1, \text{ as } \xi \to 0+. \]  
(14)

We need to prove that \( g(v) = v e^{-\frac{v}{\xi}} \) is decreasing for \( v \geq 1 \). Indeed we have that \( g'(v) = e^{-\frac{v}{\xi}} \left( 1 - \frac{\nu}{\xi} \right) \leq 0 \), by \( \xi \leq 1 \leq v \).

So that, again by (4), we get that
\[ 1 + 2 \sum_{\nu=1}^{\infty} \left( 1 + \frac{\nu}{\xi} \right) e^{-\frac{\nu}{\xi}} \leq 1 + 2 \left[ \int_{1}^{\infty} \left( 1 + \frac{\nu}{\xi} \right) e^{-\frac{\nu}{\xi}} dv + \left( 1 + \frac{1}{\xi} \right) e^{-\frac{1}{\xi}} \right] =: (\ast) \]  
(15)

Using integration by parts we have
\[ \int_{\frac{1}{\xi}}^{\infty} xe^{-x} dx = -e^{-x} (x + 1) \Big|_{\frac{1}{\xi}}^{\infty} = e^{-\frac{1}{\xi}} \left( \frac{1}{\xi} + 1 \right). \]  
(16)

Hence we get
\[ \int_{1}^{\infty} \left( 1 + \frac{\nu}{\xi} \right) e^{-\frac{\nu}{\xi}} dv = \int_{1}^{\infty} e^{-\frac{v}{\xi}} dv + \int_{1}^{\infty} \frac{v}{\xi} e^{-\frac{v}{\xi}} dv = \xi e^{-\frac{1}{\xi}} + \xi \int_{\frac{1}{\xi}}^{\infty} xe^{-x} dx = e^{-\frac{1}{\xi}} (2 \xi + 1). \]  
(17)

Therefore
\[ (\ast) = 1 + 2 \left[ e^{-\frac{1}{\xi}} (2 \xi + 1) + \left( 1 + \frac{1}{\xi} \right) e^{-\frac{1}{\xi}} \right] = 1 + 2 e^{-\frac{1}{\xi}} \left( 2 \xi + 2 + \frac{1}{\xi} \right). \]  
(18)

Consequently we have found that
\[ \sum_{\nu=-\infty}^{\infty} \left( 1 + \frac{|\nu|}{\xi} \right) e^{-\frac{|\nu|}{\xi}} = 1 + 2 \sum_{\nu=1}^{\infty} \left( 1 + \frac{\nu}{\xi} \right) e^{-\frac{\nu}{\xi}} \leq 1 + 2 e^{-\frac{1}{\xi}} \left( 2 \xi + 2 + \frac{1}{\xi} \right) \text{ (finite)} \to 1, \text{ as } \xi \to 0+. \]  
(19)
Finally we observe

\[
(P^*_\xi f)(x) - f(x) = \sum_{\nu=-\infty}^{\infty} \frac{f(x+\nu) - f(x)}{e^{-\nu \xi}}. \tag{20}
\]

So that

\[
\left| (P^*_\xi f)(x) - f(x) \right| \leq \sum_{\nu=-\infty}^{\infty} \frac{\omega_1(f,|\nu|) e^{-\nu \xi}}{e^{-\nu \xi}} \tag{21}
\]

\[
\leq \frac{\omega_1(f,|\nu|)}{e^{-\nu \xi}} \left( \sum_{\nu=-\infty}^{\infty} \left(1 + \frac{|\nu|}{\xi} \right) e^{-\nu \xi} \right) \tag{22}
\]

by (20).

\[
\leq \omega_1(f,\xi) \left( \frac{1 + 2e^{-\frac{\xi}{2}} \left(2\xi + 2 + \frac{1}{\xi}\right)}{2\xi e^{-\frac{\xi}{2}} + 1} \right). \tag{23}
\]

We notice that

\[
\frac{1 + 2e^{-\frac{\xi}{2}} \left(2\xi + 2 + \frac{1}{\xi}\right)}{2\xi e^{-\frac{\xi}{2}} + 1} \to 1, \text{ as } \xi \to 0^+.
\]

We have proved

\[
\left| (P^*_\xi f)(x) - f(x) \right| \leq \omega_1(f,\xi), \tag{24}
\]

\forall x \in \mathbb{R}.

The proof now is completed. ⊓⊔

Next we prove preservation of global smoothness of \( P^*_\xi \).

**Theorem 2.2.** It holds

\[
\omega_1(P^*_\xi f, \delta) \leq \omega_1(f, \delta), \quad \forall \delta > 0. \tag{25}
\]

Inequality (25) is sharp, namely it is attained by \( f(x) = \text{identity}(x) = x \).

**Proof.** We see that

\[
(P^*_\xi f)(x) - (P^*_\xi f)(y) = \sum_{\nu=-\infty}^{\infty} \frac{f(x+\nu) - f(y+\nu)}{e^{-\nu \xi}}. \tag{26}
\]
Hence
\[
| (P_\xi^* f)(x) - (P_\xi^* f)(y) | \leq \frac{\sum_{\nu=\infty}^{\infty} |f(x+\nu) - f(y+\nu)| e^{\frac{-\nu}{\xi}}}{\sum_{\nu=\infty}^{\infty} e^{\frac{-\nu}{\xi}}} \\
\leq \frac{\sum_{\nu=\infty}^{\infty} \omega_1(f,|x-y|) e^{\frac{-\nu}{\xi}}}{\sum_{\nu=\infty}^{\infty} e^{\frac{-\nu}{\xi}}} = \omega_1(f,|x-y|).
\]
(27)

So that for any \( x, y \in \mathbb{R} \) : \( |x-y| < \delta \) we get (25).

If \( f = \text{id} \), then trivially we get
\[
(P_\xi^* \text{id})(x) - (P_\xi^* \text{id})(y) = x - y = \text{id}(x) - \text{id}(y),
\]
(28)
thus (25) is attained.

Next we study the approximation properties of discrete Gauss-Weierstrass operators.

**Theorem 2.3.** Let \( f \in C_0(\mathbb{R}) \), \( 0 < \xi \leq 1 \). Then
\[
\| W_\xi^* f - f \|_\infty \leq C(\xi) \omega_1 (f, \sqrt{\xi}),
\]
(29)
where
\[
C(\xi) := \left[ 1 + \frac{\frac{1}{\sqrt{\pi \xi}} \left( \sqrt{\xi} + 2 + \frac{e}{\sqrt{\xi}} \right)}{\left( 1 - \text{erf} \left( \frac{1}{\sqrt{\xi}} \right) \right)^2 + 1} \right].
\]
(30)

We have \( \lim_{\xi \to 0^+} C(\xi) = 1 \), and by \( \lim_{\xi \to 0^+} \omega_1 (f, \sqrt{\xi}) = 0 \), we get \( W_\xi^* \to I \) uniformly with rates, as \( \xi \to 0^+ \).

**Proof.** We notice easily that
\[
\sum_{\nu=\infty}^{\infty} e^{\frac{-\nu^2}{\xi}} = \sum_{\nu=1}^{\infty} e^{\frac{-\nu^2}{\xi}} < \sum_{\nu=1}^{\infty} \frac{\pi^2}{6} < \infty.
\]
(31)

So we can write
\[
\sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}} = 2 \sum_{\nu=1}^{\infty} e^{\frac{-\nu^2}{\xi}} + 1 < \frac{\pi^2}{3} + 1.
\]
(32)

Since \( e^{\frac{-\nu^2}{\xi}} \) is positive, continuous and decreasing, by [4], we get
\[
\sum_{\nu=1}^{\infty} e^{\frac{-\nu^2}{\xi}} - e^{\frac{-1}{\xi}} \leq \int_{1}^{\infty} e^{\frac{-\nu^2}{\xi}} d\nu \leq \sum_{\nu=1}^{\infty} e^{\frac{-\nu^2}{\xi}}.
\]
(33)

So that
\[
2 \int_{1}^{\infty} e^{\frac{-\nu^2}{\xi}} d\nu + 1 \leq 2 \sum_{\nu=1}^{\infty} e^{\frac{-\nu^2}{\xi}} + 1 = \sum_{\nu=-\infty}^{\infty} e^{\frac{-\nu^2}{\xi}},
\]
(34)
and 

\[ 0 < \frac{1}{\sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{2}}} \leq \frac{1}{2 \int_{1}^{\infty} e^{-\frac{\nu^2}{2}} d\nu + 1}. \]  

(35)

We know that \( \int_{0}^{\infty} e^{-\nu^2} d\nu = \frac{\sqrt{\pi}}{2}, \) and \( \text{erf} (x) := \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} dt, \) with \( \text{erf} (\infty) = 1. \)

Hence

\[ 2 \int_{1}^{\infty} e^{-\frac{\nu^2}{2}} d\nu + 1 = 2 \sqrt{\xi} \int_{1}^{\infty} e^{-(\nu^2)} d\left( \frac{\nu}{\sqrt{\xi}} \right) + 1 = \]

\[ 2 \sqrt{\xi} \int_{1}^{\infty} e^{-\nu^2} d\theta + 1 = 2 \sqrt{\xi} \left[ \int_{0}^{\infty} e^{-\nu^2} d\theta - \int_{0}^{1/\sqrt{\xi}} e^{-\nu^2} d\theta \right] + 1 \]

\[ = 2 \sqrt{\xi} \left[ \frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \text{erf} \left( \frac{1}{\sqrt{\xi}} \right) \right] + 1 = \sqrt{\pi} \xi \left( 1 - \text{erf} \left( \frac{1}{\sqrt{\xi}} \right) \right) + 1. \]

(37)

Therefore

\[ 2 \int_{1}^{\infty} e^{-\frac{\nu^2}{2}} d\nu + 1 = \sqrt{\pi} \xi \left( 1 - \text{erf} \left( \frac{1}{\sqrt{\xi}} \right) \right) + 1 \to 1, \text{ as } \xi \to 0+. \]

(38)

So we got that

\[ 0 < \frac{1}{\sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{2}}} \leq \frac{1}{\sqrt{\pi} \xi \left( 1 - \text{erf} \left( \frac{1}{\sqrt{\xi}} \right) \right) + 1} \to 1, \text{ as } \xi \to 0+. \]

(39)

Next we prove that \( g (\nu) = \nu e^{-\frac{\nu^2}{2}} \) is decreasing for \( \nu \geq 1. \) Indeed we have \( g' (\nu) = e^{-\frac{\nu^2}{2}} \left( 1 - \frac{2\nu^2}{\nu} \right) \leq 0, \) if \( 1 - \frac{2\nu^2}{\nu} \leq 0, \) if \( \nu \leq 2\nu^2, \) which is true.

So that we have (by [41])

\[ \sum_{\nu=1}^{\infty} \left( 1 + \frac{\nu}{\sqrt{\xi}} \right) e^{-\frac{\nu^2}{2}} \leq \int_{1}^{\infty} \left( 1 + \frac{\nu}{\sqrt{\xi}} \right) e^{-\frac{\nu^2}{2}} d\nu + \left( 1 + \frac{1}{\sqrt{\xi}} \right) e^{-\frac{1}{\xi}} = \]

\[ \int_{1}^{\infty} e^{-\frac{\nu^2}{2}} d\nu + \int_{1}^{\infty} \frac{\nu}{\sqrt{\xi}} e^{-\frac{\nu^2}{2}} d\nu + e^{-\frac{1}{\xi}} + e^{-\frac{1}{\sqrt{\xi}}} = \]

\[ \frac{\sqrt{\pi} \xi}{2} \left( 1 - \text{erf} \left( \frac{1}{\sqrt{\xi}} \right) \right) + \frac{\sqrt{\xi}}{2} \int_{1}^{\infty} e^{-x} dx + e^{-\frac{1}{\xi}} + e^{-\frac{1}{\sqrt{\xi}}} = \]

\[ \frac{\sqrt{\pi} \xi}{2} \left( 1 - \text{erf} \left( \frac{1}{\sqrt{\xi}} \right) \right) + \frac{\sqrt{\xi}}{2} e^{-\frac{1}{\xi}} + e^{-\frac{1}{\sqrt{\xi}}} + e^{-\frac{1}{\sqrt{\xi}}} \]

(41)

That is

\[ \sum_{\nu=1}^{\infty} \left( 1 + \frac{\nu}{\sqrt{\xi}} \right) e^{-\frac{\nu^2}{2}} \leq \frac{\sqrt{\pi} \xi}{2} \left( 1 - \text{erf} \left( \frac{1}{\sqrt{\xi}} \right) \right) + e^{-\frac{1}{\xi}} + e^{-\frac{1}{\sqrt{\xi}}} \]

(42)

(finite) \to 0, \text{ as } \xi \to 0+. \]
Since
\[ \sum_{\nu=-\infty}^{\infty} \left( 1 + \frac{|\nu|}{\sqrt{\xi}} \right) e^{-\frac{\nu^2}{\xi}} = 2 \sum_{\nu=1}^{\infty} \left( 1 + \frac{\nu}{\sqrt{\xi}} \right) e^{-\frac{\nu^2}{\xi}} + 1 < \infty, \tag{43} \]
we find
\[ \sum_{\nu=-\infty}^{\infty} \left( 1 + \frac{|\nu|}{\sqrt{\xi}} \right) e^{-\frac{\nu^2}{\xi}} \leq \sqrt{\pi \xi} \left( 1 - \text{erf} \left( \frac{1}{\sqrt{\xi}} \right) \right) + e^{-\frac{1}{\xi}} \left( \sqrt{\xi} + 2 + \frac{2}{\sqrt{\xi}} \right) + 1 \tag{44} \]
(is finite) → 1, as \( \xi \to 0^+ \).

Next we observe that
\[ \left( W_\xi^* f \right)(x) - f(x) = \frac{\sum_{\nu=-\infty}^{\infty} \left( f(x + \nu) - f(x) \right) e^{-\frac{\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}}} \tag{45} \]
Thus
\[ \left| \left( W_\xi^* f \right)(x) - f(x) \right| \leq \frac{\sum_{\nu=-\infty}^{\infty} |f(x + \nu) - f(x)| e^{-\frac{\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}}} \tag{46} \]
\[ \frac{\sum_{\nu=-\infty}^{\infty} \omega_1(f,|\nu|) e^{-\frac{\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}}} = \frac{\sum_{\nu=-\infty}^{\infty} \omega_1(f,\sqrt{\xi} \sqrt{|\nu|}) e^{-\frac{\nu^2}{\xi}}}{\sum_{\nu=-\infty}^{\infty} e^{-\frac{\nu^2}{\xi}}} \tag{47} \]
\[ \leq \omega_1(f,\sqrt{\xi}) \frac{\sqrt{\pi \xi} \left( 1 - \text{erf} \left( \frac{1}{\sqrt{\xi}} \right) \right) + e^{-\frac{1}{\xi}} \left( \sqrt{\xi} + 2 + \frac{2}{\sqrt{\xi}} \right) + 1}{\sqrt{\pi \xi} \left( 1 - \text{erf} \left( \frac{1}{\sqrt{\xi}} \right) \right) + 1} \tag{49} \]
\[ \omega_1(f,\sqrt{\xi}) \left( 1 + \frac{e^{-\frac{1}{\xi}} \left( \sqrt{\xi} + 2 + \frac{2}{\sqrt{\xi}} \right)}{\sqrt{\pi \xi} \left( 1 - \text{erf} \left( \frac{1}{\sqrt{\xi}} \right) \right) + 1} \right). \tag{49} \]
So we have proved that
\[ \left| \left( W_\xi^* f \right)(x) - f(x) \right| \leq \omega_1(f,\sqrt{\xi}) \left( 1 + \frac{e^{-\frac{1}{\xi}} \left( \sqrt{\xi} + 2 + \frac{2}{\sqrt{\xi}} \right)}{\sqrt{\pi \xi} \left( 1 - \text{erf} \left( \frac{1}{\sqrt{\xi}} \right) \right) + 1} \right), \tag{50} \]
\( \forall x \in \mathbb{R}, \text{any} \ 0 < \xi \leq 1. \)

The constant in the last inequality converges to 1, as \( \xi \to 0^+ \).

The proof of the theorem is completed.

It follows the global smoothness preservation property of \( W_\xi^* \).
Theorem 2.4. It holds

\[ \omega_1 (W^*_\xi f, \delta) \leq \omega_1 (f, \delta), \quad \forall \delta > 0. \]  

(51)

Inequality (51) is sharp, attained by \( f(x) = \text{id}(x) = x \).

Proof. We see that

\[ (W^*_\xi f) (x) - (W^*_\xi f) (y) = \sum_{\nu = -\infty}^{\infty} (f(x + \nu) - f(y + \nu)) e^{-\frac{\nu^2}{\xi^2}}, \quad \forall x, y \in \mathbb{R}. \]  

(52)

Hence

\[ |(W^*_\xi f) (x) - (W^*_\xi f) (y)| \leq \sum_{\nu = -\infty}^{\infty} |f(x + \nu) - f(y + \nu)| e^{-\frac{\nu^2}{\xi^2}} \]

\[ \leq \omega_1 (f, |x - y|) \left( \sum_{\nu = -\infty}^{\infty} e^{-\frac{\nu^2}{\xi^2}} \right) \]

(53)

proving (51). Sharpness is obvious.

Next we study the approximation properties of general discrete Poisson-Cauchy operators.

Theorem 2.5. Let \( f \in C_{\text{U}}(\mathbb{R}), \ 0 < \xi \leq 1 \). Then

\[ \|M^*_\xi f - f\| \leq D(\xi) \omega_1 (f, \xi), \]  

(54)

where

\[ D(\xi) := \left[ 1 + 4\xi^{2\alpha \beta} \left( \frac{\alpha \beta}{2\alpha \beta - 1} \right) + \xi^{2\alpha \beta - 1} \left( \frac{2\alpha \beta - 1}{\alpha \beta - 1} \right) \right]. \]  

(55)

We have \( \lim_{\xi \to 0^+} D(\xi) = 1 \), and by \( \lim_{\xi \to 0^+} \omega_1 (f, \xi) = 0 \), we get \( M^*_\xi \to I \) uniformly with rates, as \( \xi \to 0^+ \).

Proof. Here \( 0 < \xi \leq 1, \alpha \in \mathbb{N}, \beta > \frac{1}{\alpha}, x \in \mathbb{R} \). By [5], p. 397, formula 595, we have

\[ \int_0^\infty \frac{1}{(t^{2\alpha} + \xi^{2\alpha})^\beta} dt = \frac{\Gamma \left( \frac{1}{2\alpha} \right) \Gamma \left( \frac{\beta - 1}{\beta} \right)}{2\Gamma (\beta) \alpha \xi^{2\alpha \beta - 1}}. \]  

(56)

Clearly \( (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \) is decreasing, continuous and positive for \( \nu \in [1, \infty) \). Hence by [4], we get

\[ 0 < \sum_{\nu = 1}^{\infty} (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \leq (1 + \xi^{2\alpha})^{-\beta} + \int_1^\infty (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} d\nu \leq \]

\[ (1 + \xi^{2\alpha})^{-\beta} + \int_0^\infty (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} d\nu = (1 + \xi^{2\alpha})^{-\beta} \]

\[ + \frac{\Gamma \left( \frac{1}{2\alpha} \right) \Gamma \left( \frac{\beta - 1}{\beta} \right)}{2\Gamma (\beta) \alpha \xi^{2\alpha \beta - 1}} < \infty, \quad \forall \xi \in (0, 1]. \]
Consequently we find convergence of

\[
0 < S_1 := \sum_{\nu=-\infty}^{\infty} (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} = \xi^{-2\alpha\beta} + 2 \sum_{\nu=1}^{\infty} (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \leq \\
\xi^{-2\alpha\beta} + 2 (1 + \xi^{2\alpha})^{-\beta} + \frac{\Gamma \left( \frac{1}{2\alpha} \right) \Gamma \left( \beta - \frac{1}{2\alpha} \right)}{\Gamma(\beta) \alpha \xi^{2\alpha\beta-1}} < \infty, \quad \forall \xi \in (0, 1].
\]  

(58)

Similarly we have

\[
\sum_{\nu=1}^{\infty} (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \geq \int_{1}^{\infty} (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \, dv,
\]
and

\[
\sum_{\nu=-\infty}^{\infty} (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \geq \xi^{-2\alpha\beta} + 2 \int_{1}^{\infty} (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \, dv.
\]

(60)

That is

\[
0 < \frac{1}{\sum_{\nu=-\infty}^{\infty} (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta}} \leq \frac{1}{\xi^{-2\alpha\beta} + 2 \int_{1}^{\infty} (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \, dv} < \xi^{2\alpha\beta}.
\]

(61)

That

\[
0 < \frac{1}{S_1} < \xi^{2\alpha\beta} \to 0, \quad \text{as} \ \xi \to 0+.
\]

(62)

Hence

\[
\lim_{\xi \to 0+} \frac{1}{S_1} = 0.
\]

(63)

Call \(g(\nu) := \nu (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta}, \ \nu \in [1, \infty)\). We have that

\[
g'(\nu) = (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \left[ 1 - \left( \frac{2\alpha\beta \nu^{2\alpha}}{\nu^{2\alpha} + \xi^{2\alpha}} \right) \right] \leq 0,
\]

(64)

iff \(1 - \left( \frac{2\alpha\beta \nu^{2\alpha}}{\nu^{2\alpha} + \xi^{2\alpha}} \right) \leq 0\), iff \(\nu^{2\alpha} + \xi^{2\alpha} \leq 2\alpha\beta \nu^{2\alpha}\), iff \(\xi^{2\alpha} \leq \nu^{2\alpha} (2\alpha\beta - 1)\), which is true because \(2\alpha\beta - 1 \geq 1\) and \(\nu^{2\alpha} (2\alpha\beta - 1) \geq 1 \geq \xi^{2\alpha}\). That is \(g\) is decreasing, positive and continuous on \([1, \infty)\).

Hence \((1 + \frac{\nu}{\xi}) (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta}\) is decreasing, positive and continuous on \([1, \infty)\).

Thus again by [4] we derive

\[
\sum_{\nu=1}^{\infty} \left( 1 + \frac{\nu}{\xi} \right) (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \leq \\
\left( 1 + \frac{1}{\xi} \right) (1 + \xi^{2\alpha})^{-\beta} + \int_{1}^{\infty} \left( 1 + \frac{\nu}{\xi} \right) (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \, dv.
\]

(65)

We further notice that

\[
\int_{1}^{\infty} \left( 1 + \frac{\nu}{\xi} \right) (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \, dv = \\
\int_{1}^{\infty} (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \, dv + \frac{1}{\xi} \int_{1}^{\infty} \nu (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \, dv < 
\]

\[
\int_{1}^{\infty} (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \, dv + \frac{1}{\xi} \int_{1}^{\infty} \nu (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \, dv < 
\]

\[
\int_{1}^{\infty} (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \, dv + \frac{1}{\xi} \int_{1}^{\infty} \nu (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \, dv <
\]
\[ \int_1^\infty \nu^{-2\alpha \beta} d\nu + \int_1^\infty \nu^{-2\alpha \beta+1} d\nu = \left( \frac{1}{2\alpha \beta - 1} \right) + \left( \frac{1}{2\xi (\alpha \beta - 1)} \right) < \infty, \forall \xi \in (0, 1]. \] 

So that

\[ \sum_{\nu=1}^{\infty} \left( \nu + \frac{\nu}{\xi} \right) (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} < (66) \]

Consequently we obtain

\[ \sum_{\nu=1}^{\infty} \left( \nu + \frac{\nu}{\xi} \right) (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} < \infty, \forall \xi \in (0, 1]. \]

0 < \( S_2 := \sum_{\nu=-\infty}^{\infty} \left( \nu + \frac{\nu}{\xi} \right) (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} < (68) \]

and

\[ \frac{S_2}{S_1} < \xi^{2\alpha \beta} \overset{(69)}{=} 1 + \frac{\xi}{\xi^{2\alpha \beta}} \left( \xi^{2\alpha} \right)^{-\beta} < \]

\[ \left[ 1 + 4\xi^{2\alpha \beta} \left( \frac{\alpha \beta}{2\alpha \beta - 1} \right) + \xi^{2\alpha \beta} \right] \overset{(70)}{=} 1, \text{ as } \xi \to 0+. \]

Hence

\[ 0 < \lim_{\xi \to 0+} S_2 < 1. \]

Finally we have that

\[ M^*_\xi (f, x) - f (x) = \frac{\sum_{\nu=-\infty}^{\infty} (f(x + \nu) - f(x)) (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta}}{\sum_{\nu=-\infty}^{\infty} (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta}}, \]

and

\[ \left| M^*_\xi (f, x) - f (x) \right| \leq \frac{\sum_{\nu=-\infty}^{\infty} \left| f(x + \nu) - f(x) \right| (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta}}{\sum_{\nu=-\infty}^{\infty} (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta}} \leq (73) \]
\[
\sum_{\nu=-\infty}^{\infty} \omega_1(f, \nu^{\frac{\nu}{\nu}}) (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \leq \sum_{\nu=-\infty}^{\infty} (\nu^{2\alpha} + \xi^{2\alpha})^{-\beta} \omega_1(f, \nu) \leq \frac{S_2}{S_1} \omega_1(f, \nu) \leq (74)
\]

We have derived
\[
\left| M_\xi^* (f, x) - f(x) \right| \leq \left[ 1 + 4 \xi^{2\alpha} \left( \frac{\alpha\beta}{2\alpha\beta - 1} \right) + \xi^{2\alpha} - 1 \left( \frac{2\alpha\beta - 1}{\alpha\beta - 1} \right) \right] \omega_1(f, \nu), \quad (75)
\]
\forall x \in \mathbb{R}, \forall \xi \in (0, 1], proving the claim.

It follows the global smoothness preservation property of \( M_\xi^* \).

**Theorem 2.6.** It holds
\[
\omega_1(M_\xi^* f, \delta) \leq \omega_1(f, \delta), \quad \forall \delta > 0. \quad (76)
\]

*Inequality (76) is sharp, attained by \( f(x) = \text{id}(x) = x \).*

**Proof.** Similar to the proof of Theorem 2.4.

We continue with

**Theorem 2.7.** It holds
\[
\left\| \theta_\xi^* f - f \right\|_\infty \leq \left( \frac{2}{3} \pi^2 K^{2\beta} \right) \xi^{2\beta - 1} \omega_1(f, \nu) \rightarrow \theta, \text{ as } \xi \rightarrow \theta^+. \quad (77)
\]

**Proof.** Here we use a lot
\[
\sum_{\nu=1}^{\infty} \frac{1}{\nu^2} = \frac{\pi^2}{6} \quad (Euler, 1741). \quad (78)
\]

We have
\[
\theta_\xi^*(f, x) - f(x) = \frac{\xi^{2\beta} \sum_{\nu \in \mathbb{Z}-\{0\}} (f(x + \nu) - f(x)) \left( \frac{\phi(\nu)}{\nu} \right)^{2\beta}}{1 + \xi^{2\beta} \sum_{\nu \in \mathbb{Z}-\{0\}} \left( \frac{\phi(\nu)}{\nu} \right)^{2\beta}}. \quad (79)
\]

Hence
\[
\left| \theta_\xi^*(f, x) - f(x) \right| \leq \frac{\xi^{2\beta} \sum_{\nu \in \mathbb{Z}-\{0\}} \left| f(x + \nu) - f(x) \right| \left( \frac{\phi(\nu)}{\nu} \right)^{2\beta}}{1 + \xi^{2\beta} \sum_{\nu \in \mathbb{Z}-\{0\}} \left( \frac{\phi(\nu)}{\nu} \right)^{2\beta}}.
\]
\[
\frac{\xi^{2\beta} \sum_{\nu \in \mathbb{Z} - \{0\}} \omega_1 \left( f, \xi \frac{\nu}{\xi} \right) \left( \frac{\varphi(\frac{\nu}{\xi})}{\nu} \right)^{2\beta}}{1 + \xi^{2\beta} \sum_{\nu \in \mathbb{Z} - \{0\}} \left( \frac{\varphi(\frac{\nu}{\xi})}{\nu} \right)^{2\beta}} \leq \frac{\xi^{2\beta} \omega_1 \left( f, \xi \right) \sum_{\nu \in \mathbb{Z} - \{0\}} \left( 1 + \frac{\nu}{\xi} \right) \left( \frac{\varphi(\frac{\nu}{\xi})}{\nu} \right)^{2\beta}}{1 + \xi^{2\beta} \sum_{\nu \in \mathbb{Z} - \{0\}} \left( \frac{\varphi(\frac{\nu}{\xi})}{\nu} \right)^{2\beta}} \leq \frac{\xi^{2\beta} \omega_1 \left( f, \xi \right) K^{2\beta} \left( \sum_{\nu \in \mathbb{Z} - \{0\}} \left( 1 + \frac{\nu}{\xi} \right) \frac{1}{\nu^2} \right)}{1 + \xi^{2\beta} \sum_{\nu \in \mathbb{Z} - \{0\}} \left( \frac{\varphi(\frac{\nu}{\xi})}{\nu} \right)^{2\beta}} := (\ast). \tag{80}
\]

We observe that
\[
0 \leq S_2 := \sum_{\nu \in \mathbb{Z} - \{0\}} \left( \frac{\varphi(\frac{\nu}{\xi})}{\nu} \right)^{2\beta} \leq K^{2\beta} \sum_{\nu \in \mathbb{Z} - \{0\}} \frac{1}{\nu^{2\beta}} = 2K^{2\beta} \sum_{\nu=1}^{\infty} \frac{1}{\nu^{2\beta}} < 2K^{2\beta} \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} = \frac{K^{2\beta} \pi^2}{3}. \tag{82}
\]

Thus
\[
0 \leq S_2 \leq \frac{K^{2\beta} \pi^2}{3}. \tag{83}
\]

So that
\[
1 \leq 1 + \xi^{2\beta} S_2 \leq 1 + \frac{\xi^{2\beta} K^{2\beta} \pi^2}{3} < \infty, \quad \forall \xi > 0. \tag{84}
\]

That is
\[
0 < \frac{1}{1 + \xi^{2\beta} S_2} \leq 1, \tag{85}
\]

with
\[
\lim_{\xi \to 0^+} \frac{1}{1 + \xi^{2\beta} S_2} = 1. \tag{86}
\]

Consequently it holds
\[
(\ast) \leq 2\xi^{2\beta} \omega_1 \left( f, \xi \right) K^{2\beta} \left( \sum_{\nu=1}^{\infty} \left( 1 + \frac{\nu}{\xi} \right) \frac{1}{\nu^{2\beta}} \right) = 2\xi^{2\beta} \omega_1 \left( f, \xi \right) K^{2\beta} \left( \sum_{\nu=1}^{\infty} \frac{1}{\nu^{2\beta}} + \frac{1}{\xi} \sum_{\nu=1}^{\infty} \frac{1}{\nu^{2\beta - 1}} \right) \leq 2\xi^{2\beta} \omega_1 \left( f, \xi \right) K^{2\beta} \left( \sum_{\nu=1}^{\infty} \frac{1}{\nu^{2\beta}} + \frac{1}{\xi} \sum_{\nu=1}^{\infty} \frac{1}{\nu^{2}} \right) = \frac{\xi^{2\beta} \pi^2}{3} \omega_1 \left( f, \xi \right) K^{2\beta} \left( 1 + \frac{1}{\xi} \right) < \frac{2\xi^{2\beta-1} \pi^2}{3} K^{2\beta} \omega_1 \left( f, \xi \right). \tag{87}
\]
We have proved that
\[
\left| \theta_\xi^*(f, x) - f(x) \right| \leq \left( \frac{2}{3} \pi^2 K^{2\beta} \right) \xi^{2\beta - 1} \omega_1(f, \xi) \to 0, \mathrm{as} \ \xi \to 0^+. \tag{88}
\]
The proof is completed.

**Example 2.8.** In Theorem 2.7 we can take \( \varphi \) to be sine, cosine with \( K = 1 \).

**Theorem 2.9.** It holds
\[
\omega_1(\theta_\xi^* f, \delta) \leq \omega_1(f, \delta), \ \forall \ \delta > 0. \tag{89}
\]
Inequality (89) is attained by \( f = \text{id} \).

We finish by studying \( T_\xi^* \).

**Theorem 2.10.** It holds
\[
\left\| T_\xi^* f - f \right\|_\infty \leq \left( \frac{2\pi^2 A}{3} \right) \xi^{2\beta - 1} \omega_1(f, \xi) \to 0, \mathrm{as} \ \xi \to 0^+. \tag{90}
\]

**Theorem 2.11.** It holds
\[
\omega_1(T_\xi^* f, \delta) \leq \omega_1(f, \delta), \ \forall \ \delta > 0. \tag{91}
\]
Inequality (91) is attained by \( f = \text{id} \).

**Proof. of Theorem 2.10**

We have
\[
T_\xi^* (f, x) - f(x) = \frac{\sum_{\nu \in \mathbb{Z} \setminus \{0\}} f(x + \nu) - f(x) \left( \frac{\psi(\frac{\nu}{\xi})}{(\frac{\nu}{\xi})^{\beta}} \right)}{1 + \sum_{\nu \in \mathbb{Z} \setminus \{0\}} \left( \frac{\psi(\frac{\nu}{\xi})}{(\frac{\nu}{\xi})^{\beta}} \right)}. \tag{92}
\]
Thus
\[
\left| T_\xi^* (f, x) - f(x) \right| \leq \frac{\sum_{\nu \in \mathbb{Z} \setminus \{0\}} |f(x + \nu) - f(x)| \left( \frac{\psi(\frac{\nu}{\xi})}{(\frac{\nu}{\xi})^{\beta}} \right)}{1 + \sum_{\nu \in \mathbb{Z} \setminus \{0\}} \left( \frac{\psi(\frac{\nu}{\xi})}{(\frac{\nu}{\xi})^{\beta}} \right)}. \tag{93}
\]
\[
\leq \frac{\sum_{\nu \in \mathbb{Z} \setminus \{0\}} \omega_1(f, \xi |\nu| \xi) \left( \frac{\psi(\frac{\nu}{\xi})}{(\frac{\nu}{\xi})^{\beta}} \right)}{1 + \sum_{\nu \in \mathbb{Z} \setminus \{0\}} \left( \frac{\psi(\frac{\nu}{\xi})}{(\frac{\nu}{\xi})^{\beta}} \right)}. \tag{94}
\]
\[
\omega_1 (f, \xi) \sum_{\nu \in \mathbb{Z} - \{0\}} \left( 1 + \frac{|\nu|}{\xi} \right) \frac{A_{\nu \xi}}{(\frac{\nu}{\xi})^{2\beta}} \\
\leq \frac{\omega_1 (f, \xi) \sum_{\nu \in \mathbb{Z} - \{0\}} \left( 1 + \frac{|\nu|}{\xi} \right) \frac{A_{\nu \xi}}{(\frac{\nu}{\xi})^{2\beta}}}{1 + \sum_{\nu \in \mathbb{Z} - \{0\}} \left( \frac{\nu}{\xi} \right)^{2\beta}} \\
= A\omega_1 (f, \xi) \xi^{2\beta} \sum_{\nu \in \mathbb{Z} - \{0\}} \left( 1 + \frac{|\nu|}{\xi} \right) \frac{1}{\nu^{2\beta}} \\
= 2A\omega_1 (f, \xi) \xi^{2\beta} \sum_{\nu = 1}^{\infty} \left( 1 + \frac{\nu}{\xi} \right) \frac{1}{\nu^{2\beta}} \\
\leq 2A\omega_1 (f, \xi) \xi^{2\beta} \left[ \frac{\pi^2}{6} + \frac{1}{\xi^{2\beta}} \sum_{\nu = 1}^{\infty} \frac{1}{\nu^{2\beta}} \right] \\
= 2A\omega_1 (f, \xi) \xi^{2\beta} \left[ \frac{\pi^2}{6} + \frac{1}{\xi^{2\beta}} \right] \\
\leq \frac{2A\omega_1 (f, \xi) \xi^{2\beta} \left( \frac{\pi^2}{6} + \frac{1}{\xi^{2\beta}} \right)}{1 + \sum_{\nu \in \mathbb{Z} - \{0\}} \left( \frac{\nu}{\xi} \right)^{2\beta}} \leq \left( \frac{2\pi^2 A}{3} \right)\omega_1 (f, \xi) \xi^{2\beta-1} =: (\ast). 
\]

We see that
\[
1 + \sum_{\nu \in \mathbb{Z} - \{0\}} \frac{\varphi \left( \frac{\nu}{\xi} \right)}{\left( \frac{\nu}{\xi} \right)^{2\beta}} \leq 1 + A \sum_{\nu \in \mathbb{Z} - \{0\}} \frac{1}{\nu^{2\beta}} \\
= 1 + \xi^{2\beta} \sum_{\nu \in \mathbb{Z} - \{0\}} \frac{1}{\nu^{2\beta}} = 1 + 2A\xi^{2\beta} \sum_{\nu = 1}^{\infty} \frac{1}{\nu^{2\beta}} \\
< 1 + 2A\xi^{2\beta} \left( \sum_{\nu = 1}^{\infty} \frac{1}{\nu^{2}} \right) = 1 + \frac{A\xi^{2\beta} \pi^2}{3} < \infty, \quad \forall \xi > 0. 
\]

That is
\[
1 \leq 1 + \sum_{\nu \in \mathbb{Z} - \{0\}} \frac{\varphi \left( \frac{\nu}{\xi} \right)}{\left( \frac{\nu}{\xi} \right)^{2\beta}} < 1 + \frac{A\xi^{2\beta} \pi^2}{3} < \infty, \quad \forall \xi > 0. 
\]

Also \(1 + \frac{A\xi^{2\beta} \pi^2}{3} \to 1\), as \(\xi \to 0^+\), that is
\[
\lim_{\xi \to 0^+} \left( 1 + \sum_{\nu \in \mathbb{Z} - \{0\}} \frac{\varphi \left( \frac{\nu}{\xi} \right)}{\left( \frac{\nu}{\xi} \right)^{2\beta}} \right) = 1. 
\]

Furthermore we have
\[
0 < \frac{1}{1 + \sum_{\nu \in \mathbb{Z} - \{0\}} \varphi \left( \frac{\nu}{\xi} \right)} \leq 1. 
\]
Hence
\[(*) \leq \left( \frac{2\pi^2}{3} A \right) \omega_1(f, \xi) \xi^{2\beta-1}. \tag{103} \]

We have proved
\[|T_\xi^*(f, x) - f(x)| \leq \left( \frac{2\pi^2 A}{3} \right) \xi^{2\beta-1} \omega_1(f, \xi) \to 0, \quad \text{as } \xi \to 0+, \quad \forall \, x \in \mathbb{R}, \tag{104} \]
proving the claim.

**Note 2.12.** All estimates of this article are also true for \( f \in C_b(\mathbb{R}) \), continuous and bounded functions on \( \mathbb{R} \). However the convergences fail if \( f \) is not uniformly continuous.

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### References


A Girsanov formula associated to a big order pseudo-differential operator

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ABSTRACT

We give a quasi-invariance formula involved with a semi-group generated by a big order elliptic pseudo-differential operator.

RESUMEN

Entregamos una fórmula de cuasi-invarianza relacionada con un semigrupo generado por un operador seudo-diferencial elíptico de orden superior.

Keywords and Phrases: Pseudo-differential operators. Girsanov formula.

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1 Introduction

*Dedicated to professor doctor N’Guérékata for his birthday*

There are two basic tools in the theory of stochastic processes ([2], [6], [17]):

- Itô formulas.
- Quasi-invariance formulas of Girsanov type.

Roughly speaking, a semi group \( \exp[\mathbb{T}L] \) governed by a generator \( L \) whose domain is continuously densely imbedded in the space of bounded continuous functions \( C_b(\mathbb{R}^d) \) endowed with the uniform norm is represented by a stochastic process

\[
\exp[\mathbb{T}L]f(x) = \mathbb{E}[f(x_t(x))]
\]

if and only if the generator satisfies the maximum principle: \( Lf(x) \geq 0 \) if the function \( f \) reaches his maximum in \( x \). Such semi-groups are called Markov semi-groups.

There are much more semi-groups than Markov semi-groups.

Itô formula was extended for more general partial differential equations in [7], [8], [9], [10], [11], [15]. For an approach to Itô formula to generalized Wiener chaos, we refer to [13], [14].

Girsanov formula was extended in the framework of white noise analysis to bilaplacians in [13] and [14].

We refer to [16] to a review.

The object of this paper is to extend the Girsanov formula to very general semi-groups generated by general pseudo-differential operators.

2 Statement and proof of the main theorem

Let \( (x, \xi) \to a(x, \xi) \) a smooth function on \( \mathbb{R}^d \times \mathbb{R}^d \). According the terminology of [3], [4], [5] it is called a symbol. We suppose that if \( |\xi| \leq C \) the symbol is smooth with bounded derivatives at each order. If \( |\xi| > C \), we suppose that there exist a strictly positive integer \( m \) such that

\[
\sup_{x \in \mathbb{R}^d} |D_x^k D_{\xi}^{k'} a(x, \xi)| \leq C|\xi|^{2m-k'}
\]

We suppose that the symbol is elliptic:

\[
\inf_{x \in \mathbb{R}^d} |a(x, \xi)| \geq C|\xi|^{2m}
\]

We put by standard theory on pseudo-differential operators ([3], [4], [5])

\[
\hat{L}\hat{f}(x) = \int_{\mathbb{R}^d} a(x, \xi)\hat{f}(\xi) d\xi
\]
where $\xi \to \hat{f}(\xi)$ is the Fourier transform of $x \to f(x)$. He can be extended continuously on the space of smooth functions with bounded derivatives at each order. We suppose because later we consider Girsanov type formulas that $L^1 = 0$.

**Hypothesis:** We suppose that $-L$ is positive essentially self-adjoint on $L^2(\mathbb{R}^d)$.

$L$ generates a contraction semigroup $\exp[tL]$ on $L^2(\mathbb{R}^d)$. By elliptic theory,

$$\exp[tL]f(x) = \int_{\mathbb{R}^d} f(y) \mu_t(x, dy)$$

where $\mu_t(x, dy)$ is a measure on $\mathbb{R}^d$ (But not a probability measure).

We consider an operator $L^1$ on $L^2(\mathbb{R}^d)$ and we suppose that it is a pseudodifferential operator of order strictly smaller than $2m - 1$ of the type (2) and (4). He can be extended continuously on the space of smooth functions with bounded derivatives at each order. We suppose because later we consider Girsanov type formulas that $L^1 = 0$. We consider the pseudo-differential operator densely defined on $L^2(\mathbb{R}^d \times \mathbb{R})$

$$-L^{tot} = -L - L^1 \frac{\partial}{\partial u} + (-1)^m \frac{\partial^{2m}}{\partial u^{2m}}$$

By elliptic theory, it generates a semi group $\exp[tL^{tot}]$ on $L^2(\mathbb{R}^d \times \mathbb{R})$ (But not a contraction semi-group due to the perturbation term $L^1 \frac{\partial}{\partial u}$ in the total operator $L^{tot}$). The main remark is that if $f$ depends only on $u$ $L^1 \frac{\partial}{\partial u} f = 0$! By elliptic theory

$$\exp[tL^{tot}]f(x,u) = \int_{\mathbb{R}^d \times \mathbb{R}} f(y,v) \mu^{tot}_t(x,u, dy, dv)$$

where $\mu^{tot}_t$ is a measure on $\mathbb{R}^d \times \mathbb{R}$ (But not a probability measure).

We consider the operator densely defined on $L^2(\mathbb{R}^d)$

$$-L^{per} = -L - L^1$$

By elliptic theory, it generates a semi-group on $L^2(\mathbb{R}^d)$ (But not a contraction semi-group due to the perturbation term $L^1$). By elliptic theory, it generates a semi-group on $L^2(\mathbb{R}^d)$ (but not a contraction semi-group due to the perturbation term $L^1$). By elliptic theory,

$$\exp[tL^{per}]f(x) = \int_{\mathbb{R}^d} f(y) \mu^{per}_t(x, dy)$$

where $\mu^{per}_t(x, dy)$ is a measure on $\mathbb{R}^d$ (but not a probability measure).

We get

**Theorem 2.1. (Girsanov):** We have if $f$ is continuous with compact support and if we consider the Doleans-Dade exponential $\exp[u + (-1)^m t] = g(u, t)$

$$\exp[tL^{per}]f(x) = \exp[tL^{tot}]f(\cdot, g(\cdot, t))(x, 0)$$

$$\int_{\mathbb{R}^d} f(y) \mu^{tot}_t(x, dy)$$
Proof: Let us begin by doing formal computations. \( \frac{\partial}{\partial u} \) commute with \( L_{\text{tot}} \). Therefore

\[
L_{\text{tot}} \exp[tL_{\text{tot}}][f(.)g(., t)](x, u) = \exp[tL_{\text{tot}}][f(.)g(., t)](x, u) + \exp[tL_{\text{tot}}][-1]^{m+1} \frac{\partial^{2m}}{\partial u^{2m}} g(., t)(x, u) = A_1 + A_2 + A_3 \tag{11}
\]

The term \( A_3 \) is boring. This explain that we introduce \( \exp[-1]^{m}t \) in the Doelans-Dade exponential in order to remove it. Namely we consider linear semi-groups such that

\[
\exp[tL_{\text{tot}}][f(.)g(., t)](x, u) = \exp[tL_{\text{tot}}][f(.)\exp[u)][x, 0] \exp[-1]^{m}t \tag{12}
\]

Therefore \( A_3 \) disappears and

\[
\frac{\partial}{\partial t} \exp[tL_{\text{tot}}][f(.)g(., t)](x, 0) = L_{\text{per}} \exp[tL_{\text{tot}}][f(.)g(., t)](x, 0) \tag{13}
\]

The only problem in this formal comutation is that \( u \rightarrow \exp[u] \) is not bounded!. But if \( f \) is with compact support continuous

\[
\left| \exp[tL_{\text{tot}}][f(.)\exp[u]][(x, 0)] \right| \leq \int_{R^d \times R} |f(y)| \exp[v] \mu_{t}^{\text{tot}}(x, u, dy, dv) \\
\leq \left( \int_{R^d} |f(y)|^2 \mu_{t}([x, dy]) \right)^{1/2} \left( \int_{R} \exp[2u] \nu_{t}(0, dv) \right)^{1/2} \tag{14}
\]

In (14), \( \nu_{t}(u, dv) \) represents the semi group associated to \( L_{2m} = (-1)^{m+1} \frac{\partial^{2m}}{\partial u^{2m}} \). By [1], this semi-group has an heat-kernel bounded by \( C t^{-1/4m} G_{2m,a}[|u-v|] \) \( (a > 0) \) where

\[
G_{2m,a}(u) = \exp[-au^{2m/2m-1}] \tag{15}
\]

This inequality justifies the formal considerations above!

\( \diamond \)

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References


Existence of Entire Solutions for Quasilinear Elliptic Systems under Keller-Osserman Condition

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ABSTRACT

In this paper, we study the existence of entire solutions for the following elliptic system

\[ \Delta_m u = p(x)f(v), \Delta_l v = q(x)g(u), \ x \in \mathbb{R}^N, \]

where \( 1 < m, l < \infty \), \( f, g \) are continuous and non-decreasing on \( [0, \infty) \), satisfy \( f(t) > 0, g(t) > 0 \) for all \( t > 0 \) and the Keller-Osserman condition. We establish conditions on \( p \) and \( q \) that are necessary for the existence of positive solutions, bounded and unbounded, of the given equation.

RESUMEN

En este artículo estudiamos la existencia de soluciones enteras para el siguiente sistema elíptico

\[ \Delta_m u = p(x)f(v), \Delta_l v = q(x)g(u), \ x \in \mathbb{R}^N, \]

donde \( 1 < m, l < \infty \), \( f, g \) son continuas y no decrecientes en \( [0, \infty) \), satisfaciendo \( f(t) > 0, g(t) > 0 \) para todo \( t > 0 \) y la condición de Keller-Osserman. Establecemos condiciones sobre \( p \) y \( q \) que son necesarias para la existencia de soluciones positivas, acotadas y no acotadas de la ecuación dada.

Keywords and Phrases: quasi-linear elliptic system; sub/super-solution; large solution; existence.

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1 Introduction

In this paper, we investigate the following quasilinear elliptic system

\[
\begin{align*}
\triangle_m u &= p(x)f(v), \quad x \in \mathbb{R}^N, \\
\triangle_l v &= q(x)g(u), \quad x \in \mathbb{R}^N.
\end{align*}
\]

where \(1 < m, l < \infty\), \(N \geq \max\{m, l\} + 1\), \(\triangle_m > \text{div}(|\nabla \cdot |^{m-2}\nabla \cdot )\). Denote \(d = \min\{m, l\}\), and see that \(d > 1\). By an entire large solution \((u, v)\), we mean a pair of functions \(u, v \in C^1(\mathbb{R}^N)\) that satisfies (1.1) at every point of \(\mathbb{R}^N\) and

\[
\lim_{|x| \to \infty} u(x) = \lim_{|x| \to \infty} v(x) = \infty.
\]

First, we introduce the assumptions below:

(H1) \(p, q : \mathbb{R}^N \to [0, \infty)\) and \(f, g : [0, \infty) \to [0, \infty)\) are continuous and nontrival functions;

(H2) \(f\) and \(g\) are nondecreasing on \([0, \infty)\) and \(f(t) > 0, g(t) > 0\) for all \(t > 0\);

(H3) \(H(\infty) = \lim_{r \to \infty} H(r) = \infty\), where

\[
H(r) = \int_c^r \frac{dt}{t^{\frac{1}{\sqrt{F(t)}+G(t)}}}, \quad r \geq c > 0; \quad F(t) = \int_0^t f(s) ds, \quad G(t) = \int_0^t g(s) ds.
\]

and \(c\) is a positive constant. Notice that \(H'(r) = \frac{1}{\sqrt{F(r)}+G(r)} > 0, \forall r > c\), so \(H\) has the inverse function \(H^{-1}\) on \([0, \infty)\). Denote

\[
\phi_1(r) := \max_{|x|=r} p(x), \quad \phi_2(r) := \min_{|x|=r} p(x),
\]

\[
\psi_1(r) := \max_{|x|=r} q(x), \quad \psi_2(r) := \min_{|x|=r} q(x).
\]

Since 1980s, many important results have been obtained for quasilinear elliptic systems. We will introduce some results in the following. Existence and non-existence of solutions of the quasilinear elliptic system

\[
\begin{align*}
\text{div}(|\nabla u|^{m-2}\nabla u) + f(u, v) &= 0, \quad x \in \mathbb{R}^N, \\
\text{div}(|\nabla v|^{l-2}\nabla v) + g(u, v) &= 0, \quad x \in \mathbb{R}^N
\end{align*}
\]

has gained much attention recently. See, for example, \([3, 4, 10, 15, 19, 21, 22]\).

When \(p = q = 2\), system (1.3) becomes

\[
\begin{align*}
\triangle u + f(u, v) &= 0, \quad x \in \mathbb{R}^N, \\
\triangle v + g(u, v) &= 0, \quad x \in \mathbb{R}^N
\end{align*}
\]

for which the existence and the non-existence of positive solutions and positive boundary blow-up solutions have been investigated extensively. We list here, for example, \([1, 2, 5, 6, 12-14, 16]\) and refer to the references therein.
When \( p = q = 2, f = -a(|x|)v^\alpha, g = -b(|x|)u^\beta \), system (1.3) becomes
\[
\begin{align*}
\Delta u &= a(|x|)v^\alpha, \quad x \in \mathbb{R}^N \\
\Delta v &= b(|x|)u^\beta, \quad x \in \mathbb{R}^N
\end{align*}
\] (1.4)
for which existence results for positive boundary blow-up solutions can be found in a recent paper by Lair and Wood [12]. Lair and Wood established that all positive entire radial solutions of (1.4) are boundary blow-up provided that
\[
\int_0^\infty ta(t)dt = \infty, \quad \int_0^\infty tb(t)dt = \infty.
\]
If, on the other hand
\[
\int_0^\infty ta(t)dt < \infty, \quad \int_0^\infty tb(t) < \infty,
\]
then all positive entire radial solutions of (1.4) are bounded.

F. Cirstea and V.D. Radulescu [1], extended the above results to a larger class of systems
\[
\begin{align*}
\Delta u &= a(|x|)g(v), \quad x \in \mathbb{R}^N \\
\Delta v &= b(|x|)f(u), \quad x \in \mathbb{R}^N
\end{align*}
\] (1.5)
In recent years, Zhijun Zhang et al. [23] studied the following semilinear elliptic systems
\[
\begin{align*}
\Delta u &= p(x)f(v), \quad x \in \mathbb{R}^N, (N \geq 3), \\
\Delta v &= q(x)g(u), \quad x \in \mathbb{R}^N.
\end{align*}
\] (1.5)
They obtained the existence and nonexistence of solutions for (1.5) by considering a set of hypotheses on \( p, q, f \) and \( g \).

Z.D. Yang [19], extended the above results to a class of systems
\[
\begin{align*}
\text{div}(|\nabla u|^{m-2}\nabla u) &= a(|x|)g(v), \quad x \in \mathbb{R}^N, \\
\text{div}(|\nabla v|^{l-2}\nabla v) &= b(|x|)f(u), \quad x \in \mathbb{R}^N.
\end{align*}
\]
Motivated by the results of the papers [19-23], in this paper, we consider the quasilinear elliptic system (1.1). We modify the method developed by Zhang et al. [23] and extend partial results of [23] to a quasilinear elliptic system (1.1).

\section{Main Results}

In order to establish our main result, we introduce the following hypotheses:

(H4) \( r^{N-1}(\phi_1(r) + \psi_1(r)) \) is nondecreasing for large \( r \);
(H5) there exists a positive constant \( \varepsilon \) such that
\[
\int_{0}^{\infty} t^{\frac{1-\varepsilon}{m}} (\phi_1(t) + \psi_1(t)) \frac{1}{m-1} dt < \infty,
\]
and
\[
\int_{0}^{\infty} t^{\frac{1-\varepsilon}{l}} (\phi_1(t) + \psi_1(t)) \frac{1}{l-1} dt < \infty.
\]

Our main results are as the following:

**Theorem 1.** Under the hypotheses (H1)-(H5), equation (1.1) has a positive entire bounded solution \((u, v)\).

From the above theorem, we get the following corollary

**Corollary 1.** Suppose that \( p \) and \( q \) are spherically symmetric (i.e. \( p(x) = p(|x|), q(x) = q(|x|) \)). Under hypotheses (H1)-(H3), (1.1) has one positive solution \((u, v)\).

Suppose further that \( P(\infty) = Q(\infty) = \infty \), where
\[
P(\infty) := \lim_{r \to \infty} P(r), P(r) := \int_{0}^{r} (t^{1-N} \int_{0}^{t} s^{N-1} p(s) ds) \frac{1}{m-1} dt, r \geq 0;
\]
\[
Q(\infty) := \lim_{r \to \infty} Q(r), Q(r) := \int_{0}^{r} (t^{1-N} \int_{0}^{t} s^{N-1} q(s) ds) \frac{1}{l-1} dt, r \geq 0.
\]

Then every positive radial entire solution \((u, v)\) of (1.1) is large and satisfies
\[
u(r) \geq u(0) + f(v(0)) P(r), v(r) \geq v(0) + g(u(0)) Q(r), \quad \forall r \geq 0.
\]

**Corollary 2.** Under the assumption (H1)-(H4), if (1.1) has a non-negative radial entire large solution, then at least one of the following two equations hold:
\[
\int_{0}^{\infty} r^{\frac{1-\varepsilon}{m}} (p(r) + q(r)) \frac{1}{m-1} dr = \infty, \quad \forall \varepsilon > 0.
\]
\[
\int_{0}^{\infty} r^{\frac{1-\varepsilon}{l}} (p(r) + q(r)) \frac{1}{l-1} dr = \infty, \quad \forall \varepsilon > 0.
\]

**Remark 1.** By (H1) and (H3), we have
\[
\int_{a}^{\infty} \frac{ds}{\sqrt{F(s)}} = \int_{a}^{\infty} \frac{ds}{\sqrt{G(s)}} = \infty.
\]

**Remark 2.** When \( 2 \leq d < \infty \), \( \int_{0}^{\infty} r^{\frac{1}{d}} (p(x) + q(x)) \frac{1}{d} dr = \infty \) implies
\[
\int_{0}^{\infty} (t^{1-N} \int_{0}^{t} s^{N-1} (p(x) + q(x)) ds) \frac{1}{d} dt = \infty.
\]
Remark 3. If \( \int_0^\infty \frac{ds}{\sqrt{F(s)}} < \infty \), then \( \int_0^\infty \frac{dt}{(f(t))^{s+1}} < \infty \). In other words, if \( \int_0^\infty \frac{dt}{(f(t))^{s+1}} = \infty \), then \( \int_0^\infty \frac{ds}{\sqrt{F(s)}} = \infty \).

Proof. We only need to prove
\[
\frac{(f(t))^{s+1}}{s} \geq \delta^d
\]
for all \( \delta > 0 \). Then \( F(s) \equiv \int_0^s f(t)dt \leq sf(s) \leq \frac{\int_0^s f(t)^{s+1}dt}{\delta^d} \), and \( (F(s))^{\frac{1}{s+1}} \geq \frac{\delta}{(f(s))^{\frac{1}{s+1}}} \). We suppose that (1.6) is not true, then \( \exists \) an increasing sequence \( \{s_j\} \), \( \lim_{j \to \infty} s_j = \infty \) such that \( \frac{(f(s_j))^{\frac{1}{s_j}}}{s_j} < \frac{1}{j} \), which equals to \( f(s_j) \leq \left( \frac{s_j}{j} \right)^{d-1} \), then \( f(s_j) = j^{-\frac{d-1}{d}} \). Since \( f \) is nondecreasing, we get \( f(s) \leq f(s_j) \) for all \( s \in [0, s_j] \), so \( F(s) \leq sf(s) \leq sf(s_j) \) for all \( s \in [0, s_j] \), and
\[
\int_{s_1}^{s_j} (F(s))^{\frac{1}{s+1}}ds \geq \int_{s_1}^{s_j} (sf(s_j))^{\frac{1}{s+1}}ds \geq j^{-\frac{d-1}{d}}s^{-\frac{1}{s}}ds = j^{-\frac{d-1}{d}}(1 - (\frac{s_1}{s_j})^{1-\frac{1}{s}}) \to \infty.
\]
This is a contradiction.

In order to prove the Theorem 1, we give the following lemma.

Lemma 1. For any nonnegative \( a \) and \( b \), we have
\[
(a + b)^\alpha \leq a^\alpha + b^\alpha, \quad \alpha \in (0, 1]
\]
\[
(a + b)^\beta \leq 2^{\beta-1}(a^\beta + b^\beta), \quad \beta \in [1, \infty]
\]

Proof of Theorem 1. First, we have to find a pair of super-solution, \( (u, v) \) and sub-solution, \( (\bar{u}, \bar{v}) \), which satisfy \( u \leq \bar{u} \) and \( v \leq \bar{v} \). Consider the following system of integral equation:
\[
\begin{align*}
\bar{u}(r) &= \beta + \int_0^r (t^{1-N}) \int_0^1 s^{N-1} \phi_1(s)f(v(s))ds \frac{1}{\bar{v}}dt, \quad r \geq 0 \\
\bar{v}(r) &= \beta + \int_0^r (t^{1-N}) \int_0^1 s^{N-1} \psi_1(s)g(\bar{u}(s))ds \frac{1}{\bar{v}}dt, \quad r \geq 0
\end{align*}
\]
where \( \beta \geq c > 0 \), \( c \) is in (H3). Let \( \{\bar{u}_k\}_{k \geq 0} \) and \( \{\bar{v}_k\}_{k \geq 1} \) be the sequence of positive continuous functions defined on \([0, \infty)\) by \( \bar{v}_0 = \beta \),
\[
\begin{align*}
\bar{u}_k(r) &= \beta + \int_0^r (t^{1-N}) \int_0^1 s^{N-1} \phi_1(s)f(\bar{v}_{k-1}(s))ds \frac{1}{\bar{v}_k}dt, \quad r \geq 0 \\
\bar{v}_k(r) &= \beta + \int_0^r (t^{1-N}) \int_0^1 s^{N-1} \psi_1(s)g(\bar{u}_{k-1}(s))ds \frac{1}{\bar{v}_k}dt, \quad r \geq 0
\end{align*}
\]
Then, $v_0 \leq v_1$, $u_k(r) \geq \beta$, and $v_k(r) \geq \beta$ for all $r \geq 0$, $k \in \mathbb{N}$. Using the non-decreasing property of $f$ and $g$, we get $u_1(r) \leq u_2(r)$ for all $r \geq 0$, then $v_1(r) \leq v_2(r)$ for all $r \geq 0$. Continuing this line, we obtain that the sequence $\{u_k\}$ and $\{v_k\}$ are increasing with respect to $k$ for $r \in [0, \infty)$. Besides,

$$u_k'(r) = (r^{1-N} \int_0^r s^{N-1} \phi_1(s)f(\psi_{k-1}(s))ds)^{1/\alpha} \geq 0,$$

$$v_k'(r) = (r^{1-N} \int_0^r s^{N-1} \psi_1(s)g(\psi_{k-1}(s))ds)^{1/\alpha} \geq 0,$$

for each $r > 0$, and

$$[r^{N-1}|u_k'|^{m-2}u_k']' = r^{N-1} \phi_1(r)f(\psi_{k-1}(r)) \leq r^{N-1} \phi_1(r)f(v_k(r))$$

(3)

let

$$\Theta(r) = \max_{0 \leq t \leq r} (\phi_1(t) + \psi_1(t)),$$

using this and the fact that $u_k' \geq 0$, we note that (3) yields

$$((u_k'(r))^{m-1})' \leq \Theta(r)f(v_k(r)),$$

Multiply this by $u_k'$ and integrate to get

$$(u_k'(r))^m \leq \frac{m}{m-1} \Theta(r) \int_2^\beta (f(s) + g(s))ds$$

In the same way,

$$(v_k'(r))^l \leq \frac{1}{l-1} \Theta(r) \int_2^\beta (f(s) + g(s))ds$$

Then from the inequality $(u_k' + v_k')^d \leq 2d-1((u_k')^d + (v_k')^d)$, where $d = \min(m, l)$, and the above two inequalities, we get

$$u_k' + v_k' \leq 2d-1((u_k')^d + (v_k')^d)$$

$$\leq 2d-1((u_k')^m + (v_k')^l + 1)$$

$$\leq 2d-1\left(\frac{d}{d-1} \Theta(r) \int_2^\beta (f(s) + g(s))ds + 1\right)$$

$$\leq 2d-1\left(\frac{d}{d-1} \Theta(r)(F(u + v) + G(u + v)) + 1\right)$$

(4)

which yields

$$u_k' + v_k' \leq 2d-1\left(\frac{d}{d-1} \Theta(r)(F(u_k + v_k) + G(u_k + v_k)) + 1\right)^{\frac{1}{d}}$$

$$\leq \left(\frac{2d-1}{d-1} \Theta(r)(F(u_k + v_k) + G(u_k + v_k))^{\frac{1}{d}} + 2d-1\right)^{\frac{d}{d-1}}$$

(5)
Integrating the above inequality, we get
\[
\int_0^r \frac{u_k'(t) + v_k'(t)}{F(u_k(t) + v_k(t)) + G(u_k(t) + v_k(t))} \, dt = \int_{\nu_k(r) + u_k(r)} \frac{d\tau}{\sqrt{F(\tau)} + G(\tau)} \\
\leq \int_0^r \left( \sqrt{\frac{2^{d-1}d\Theta(t)}{d-1} + C} \right) dt
\]
where \( C = \frac{2^{d-1}}{\Gamma(2\beta) + \Gamma(2\beta)} \). We can easily get
\[
H(u_k(r) + v_k(r)) \leq H(2\beta) + \int_0^r \left( \sqrt{\frac{2^{d-1}d\Theta(t)}{d-1} + C} \right) dt
\]
As we know that \( H^{-1} \) is increasing on \([0, \infty)\), so
\[
u_k(r) \leq H^{-1}(H(2\beta) + \int_0^r \left( \sqrt{\frac{2^{d-1}d\Theta(t)}{d-1} + C} \right) dt, \quad \forall r \geq 0
\]
Following by the definition of \( u_k(r) \) and \( v_k(r) \) and (H3), we get that the sequence \( \{u_k\} \) and \( \{v_k\} \) are bounded and equi-continuous on \([0, C_0]\) for arbitrary \( C_0 > 0 \). By Arzela-Ascoli theorem, \( \{u_k\} \) and \( \{v_k\} \) have subsequence converging uniformly to \( u \) and \( v \) on \([0, C_0]\). By the arbitrariness of \( C_0 \), we see that \( (u, v) \) is a positive entire solution of
\[
\Delta_m u = \phi_1(r)f(v), \quad x \in \mathbb{R}^N \\
\Delta_1 v = \psi_1(r)g(u), \quad x \in \mathbb{R}^N
\]
Then, we take conclusion that \( (u, v) \) is a positive entire sub-solution of (1.1).

In order to prove \((u, v)\) is bounded, choosing \( R > 0 \), so that \( r^{d(N-1)}(\phi_1(r) + \psi_1(r)) \) is non-decreasing on \([R, \infty)\) and \( u(r) > 0, v(r) > 0\). This is possible because of (H4). Since \((u, v)\) satisfies
\[
(r^{N-1}(u')^{m-1})' = r^{N-1}\phi_1(r)f(v(r)), \\
(r^{N-1}(v')^{m-1})' = r^{N-1}\psi_1(r)g(u(r)),
\]
\( u'(r) \geq 0 \) and \( v'(r) \geq 0 \) for \( r \geq 0 \), and (H2) hold, multiplying (8) and (9) by \( u' \) and \( v' \), respectively, and integrating from \( R \) to \( r \). Take (8) as an example,
\[
\int_R^r (s^{N-1}(u')^{m-1})'u'(s)ds = \int_R^r s^{N-1}\phi_1(s)f(v(s))u'(s)ds,
\]
which implies that
\[
m-1 r^{N-1}(u'(r))^m - m^{-1} R^{N-1}(u'(R))^m + N-1 m \int_R^r s^{N-2}(u'(s))^m ds = \int_R^r s^{N-1}\phi_1(s)f(v(s))u'(s)ds
\]
It follows that
\[
r^{N-1}(u'(r))^m \leq R^{N-1}(u'(R))^m + \frac{m}{m-1} \int_r^R s^{N-1} \phi_1(s) f(y(s)) u'(s) ds
\]

Using the monotonicity of $t^{N-1}(\phi_1(t) + \psi_1(t))$ for $t \geq 0$, we get
\[
r^{N-1}(u'(r))^m \leq \bar{C} + \frac{m}{m-1} R^{N-1}(\phi_1(R) + \psi_1(R))(F(u(r) + y(r)) + G(u(r) + y(r))
\]
for $r > R$, where $\bar{C} = R^{N-1}(u'(R))^m + R^{N-1}(y'(R))^l$.

which yields
\[
u'(r) \leq \sqrt[1-N]{\frac{m}{m-1} \int_r^R s^{N-1} \phi_1(s) f(y(s)) u'(s) ds}
\]

So
\[
u'(r) + y'(r) \leq C_1 \left( r^{\frac{1-N}{m-1}} + r^{\frac{1-N}{m-1}} \right) + \left( \sqrt[1-N]{\frac{m}{m-1} (\phi_1(r) + \psi_1(r))} + \sqrt[1-N]{\frac{1}{1-1} (\phi_1(r) + \psi_1(r))} \right) (2(F(u + y) + G(u + y)))^{\frac{1}{m-1}} + 1
\]

and
\[
\frac{d}{dr} \left( \frac{\nu(r) + y(r)}{\nu(r) + y(r)} \right) \frac{d}{d\tau} \left( \frac{\nu(r) + y(r)}{\nu(r) + y(r)} \right)
\]
\[
\leq C_1 \left( r^{\frac{1-N}{m-1}} + r^{\frac{1-N}{m-1}} \right) (F(u + y) + G(u + y))^{-\frac{1}{m-1}} + h(r) (2(F(u + y) + G(u + y))^{-\frac{1}{m-1}})
\]

where $h(r) = \sqrt[1-N]{\frac{m}{m-1} (\phi_1(r) + \psi_1(r))} + \sqrt[1-N]{\frac{1}{1-1} (\phi_1(r) + \psi_1(r))}$. We notice the fact that
\[
F(u(r) + y(r)) + G(u(r) + y(r)) \geq F(u(R) + y(R)) + G(u(R) + y(R)) = C_2
\]
for all $r \geq R$, and
\[
\sqrt[1-N]{\frac{m}{m-1} (\phi_1(r) + \psi_1(r))} \leq \frac{m}{m-1} r^{1+N} (\phi_1(r) + \psi_1(r)) r^{-1-N}
\]

Using Young’s inequality, we get
\[
\sqrt[1-N]{\frac{m}{m-1} (\phi_1(r) + \psi_1(r))} \leq \frac{1}{m-1} r^{-1-N} + r^{1+N} (\phi_1(r) + \psi_1(r)) r^{-1-N} \quad \text{for } \varepsilon > 0.
\]

In the same way,
\[
\sqrt[1-N]{\frac{1}{1-1} (\phi_1(r) + \psi_1(r))} \leq \frac{1}{1-1} r^{-1-N} + r^{1+N} (\phi_1(r) + \psi_1(r)) r^{-1-N} \quad \text{for } \varepsilon > 0.
\]
Then integrate \([10]\) from \(R\) to \(r\), \(r \geq R\),

\[
H(u(r) + v(r)) - H(u(R) + v(R)) \leq C_3 + C_4(\frac{1}{m - 1} + \frac{1}{l - 1}) \frac{R^\alpha}{\epsilon} + \int_R^r t^{\frac{1}{m-1}} (\phi_1 + \psi_1)^\frac{1}{m-1} dt + \int_R^r t^{\frac{1}{l-1}} (\phi_1 + \psi_1)^\frac{1}{l-1} dt
\]

where \(C_3 = \sqrt{\frac{C_2}{C_1}} \left( \frac{mR}{N - m - 1} + \frac{1}{m - 1} \right)\), \(C_4 = 2 + \left( F(2R) + G(2R) \right)^{\frac{1}{2}}\).

From (H5), we know

\[
\int_R^r t^{\frac{1}{m-1}} (\phi_1 + \psi_1)^\frac{1}{m-1} dt < \infty,
\]

and

\[
\int_R^r t^{\frac{1}{l-1}} (\phi_1 + \psi_1)^\frac{1}{l-1} dt < \infty.
\]

So

\[
H(u(r) + v)(r) < \infty
\]

Letting \(r \to \infty\), since \(H\) satisfies (H3), we find that \((u, v)\) is bounded.

By now, we have find a pair of bounded sub-solution to (1.1). We still have to find \((\bar{u}, \bar{v})\), which is a bounded super-solution of (1.1), and \(u(r) \leq \bar{u}(r)\), \(v(r) \leq \bar{v}(r)\) for all \(r \geq 0\). Actually, since \((u, v)\) is nondecreasing and bounded, we have

\[
\lim_{r \to \infty} u(r) = M_1 > 0, \quad \lim_{r \to \infty} v(r) = M_2 > 0.
\]

Let \(\tilde{u}(0) = \tilde{v}(0) = \max\{M_1, M_2\}\), \(\tilde{u}'(0) = \tilde{v}'(0) = 0\), then, the following system

\[
\Delta_m \tilde{u}(x) = \phi_2(r) f(\tilde{v}(r)) \quad r > 0
\]

\[
\Delta_l \tilde{v}(x) = \psi_2(r) g(\bar{u}(r)) \quad r > 0
\]

has a bounded solution \((\tilde{u}, \tilde{v})\) by the same argument, and it is a supersolution for (1.1). From the above process, we get conclusion that

\[
u(r) \leq M_1 \leq \bar{u}(r), \quad v(r) \leq M_2 \leq \bar{v}(r). \quad \forall r \geq 0.
\]

The standard super-sub solution principle [18,20] implies that (1.1) has a bounded solution \((u, v)\) satisfying \(u(x) \leq u(x) \leq \tilde{u}\) and \(v(x) \leq v(x) \leq \tilde{v}\) on \(\mathbf{R}^N\), which is the desired solution. This completes the proof.

### 3 Conclusion

The boundary value quasilinear differential equation systems (1.1) are mathematical models occurring in the studies of the \(m\)-Laplace equation, generalized reaction-diffusion theory, non-Newtonian fluid theory, and the turbulent flow of a gas in porous medium. When \(m \neq 2\), the
problem becomes more complicated since certain nice properties inherent to the case \( m = 2 \) seem to be lost or at least difficult to verify. The main differences between \( m = 2 \) and \( m \neq 2 \) can be founded in [8,9]. When \( m = 2 \), it is well known that all the positive solutions in \( C^2(B_R) \) of the problem

\[
\begin{cases}
\triangle u + f(u) = 0 \text{ in } B_R \\
u(x) = 0 \text{ on } \partial B_R
\end{cases}
\]

are radially symmetric solutions for very general \( f \)(see [7]). Unfortunately, this result does not apply to the case \( m \neq 2 \). Kichenassary and Smoller showed that there exist many positive nonradial solutions of the above problem for some \( f \)(see [11]). The major stumbling block in the case of \( m \neq 2 \) is that certain nice features inherent to the case \( m = 2 \) seem to be lost or at least difficult to verify. In this paper, we first give some necessary preliminary knowledge. Secondly, we further study the existence of positive solutions to problem (1.1) which the right hand side functions are more general based on the method of sub-supersolution.

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Planar Pseudo-almost Limit Cycles and Applications to solitary Waves

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ABSTRACT
We investigate the existence of pseudo-almost limit cycles, a new class of non-periodicity at the interface of the theories of limit cycles and pseudo-almost periodicity. We determine the conditions of existence for several systems including some pseudo-almost periodic perturbations of the harmonic oscillator and the renowned Liénard systems. We apply to derive the existence of pseudo-almost periodic solitary waves by perturbing first then transforming some hyperbolic and parabolic partial differential equations to Liénard-type equations. Included also are open questions on the co-existence of limit cycles and strictly pseudo-almost periodic limit cycles partitioning the phase space, and the existence of isochronous pseudo-almost limit cycles.

RESUMEN
Investigamos la existencia de ciclos seudo-casi límites, una nueva clase de no-periodicidad en la interfaz de las teorías de ciclos límites y seudo-casi periodicidad. Determinamos condiciones de existencia de muchos sistemas, incluyendo algunas perturbaciones seudo-casi periódicas del oscilador armónico y los sistemas de Liénard. Aplicamos las condiciones para derivar la existencia de ondas solitarias seudo-casi periódicas, primero perturbando y luego transformando algunas ecuaciones diferenciales parciales hiperbólicas y parabólicas a ecuaciones del tipo Liénard. También se incluyen preguntas abiertas sobre la co-existencia de ciclos límite y estrictamente pseudo-casi periódicos ciclos límite de partición del espacio de fases, y la existencia de isócrono pseudo-casi ciclos límite.


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1 Introduction

Limit cycles are used to model the dynamical state of self-sustained oscillations found very often in biology, chemistry, mechanics, electronics, fluid dynamics, etc. See for example [2, 10, 18, 26]. They often arise in many physical systems around a state at which energy generation and dissipation balance. One of the most important limit cycles of our lives is the heartbeat. A spectacular example is the Tacoma Narrows Bridge[1] and its 1940 dramatic collapse, where the limit cycle drew its energy from the wind and involved torsional oscillations of the roadbed. In Robotics the stable gait to which the repeated dynamic walking pattern converges is modeled as a stable limit cycle, stability easily lost to even small disturbances, evidence of a narrow basin of attracting of the limit cycle.

Planar limit cycles were defined by Poincaré[2] in the famous paper \( \text{Mémoire sur les courbes définies par une équation différentielle} \) [22], using his so-called Method of sections. However much attention in this century has been drawn to the determination of the number, amplitude and configuration of limit cycles in a general nonlinear system, which is still an unsolved problem. This is part of the so-called Hilbert’s 16th problem[3]. A weakened version[4] by Arnold called the tangential Hilbert’s problem, concerns the bound on the number of limit cycles which can bifurcate from a first-order perturbation of a Hamiltonian system.[3, 9, 13, 14, 17]

The possibility of a limit cycle on a plane or a two-dimensional manifold is restricted to nonlinear dynamical systems, due to the fact that, for linear systems, \( kx(t) \) is also a solution for any constant \( k \) if \( x(t) \) is a solution. Therefore the phase space will contain an infinite number of closed trajectories encircling the origin, with none of them isolated. Conservative and gradient systems do not have limit cycles, but these systems may exhibit almost or pseudo-almost limit cycles. The most common techniques for predicting the absence or existence of periodicity and limit cycles include the Index Theory, Dulac’s Criterion, Poincaré-Bendixson Test, Perturbation and Bifurcation theory, Configuration of limit cycles, the Toroidal Principle. These concepts and related examples could be found in [2, 5, 6, 9, 10, 13, 18, 25]. The nonlinear character of isolated periodic oscillations renders their detection and construction challenging. In mechanical terms the appraisal of the regions of the phase plane where energy loss and energy gain occur might reveal a limit cycle.

Let us emphasize that even though in most studies periodicity has been illustrated more frequently, almost and pseudo-almost periodic oscillations or waves actually occur much more

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1 A wealth of information including historical and anecdotal facts could be found in http://en.wikipedia.org/wiki/Tacoma-Narrows-Bridge(1940)
2 Jules Henri Poincaré has excelled in all fields of knowledge and is often described as a polymath or The Last Universalist. The famous Poincaré conjecture named after him was finally solved in 2002-2003 by Grigori Perelman who turned down the related prize of $1,000,000!
3 Stated in 1900, it was only in 1987 that Écalle and Ilyashenko proved independently the finiteness of that number using the compactification of the phase space to Poincaré disk
4 The number of limit cycles in a small perturbation of a polynomial Hamiltonian system is given by the number of zeroes of Abelian integrals at least far from polycycles.
frequently than periodic ones. For instance, in the simplest model of harmonic oscillator or mathematical pendulum, as well as for the one-dimensional wave equation, diverse kinds of oscillatory trajectories can be displayed, both periodic and more generally non-periodic.

The theory of almost periodic functions introduced by H. Bohr [6] in the 1920s and extended to pseudo-almost periodicity by Zhang [27] in the 1990s is also connected with problems in differential equations, stability theory, dynamical systems, partial differential equations or equations in Banach spaces. There are several results concerning the existence and uniqueness of almost and pseudo-almost periodic solutions for first-order differential equations, e.g., in [7, 11, 12, 15, 20, 21, 23, 24, 27]. But the authors usually derived their results from the existence of bounded solutions.

We extend the theory of limit cycles and pseudo-almost periodicity to that of pseudo-almost limit cycles, isolated pseudo-almost periodic orbits, and we investigate in the current and future work the usual questions of conditions of existence and uniqueness, stability, bifurcation and perturbation, the coexistence of limit cycles and strictly pseudo-almost limit cycles. We also introduce the idea of isochronous pseudo-almost limit cycles and pseudo-almost isochrons.

Section 2 overviews the theory of limit cycles recalling the definitions and presenting some classic and concrete examples relevant to our study. In section 3, we develop the concept of pseudo-almost limit cycle, its properties, several illustrative examples including the so-called linear pseudo-center, and existence theorems in the case of the well-known Liénard systems. Section 4 shows the applications of the existence theorems for Liénard systems to obtain pseudo-almost periodic solitary waves for some hyperbolic and parabolic partial differential equations. Finally in section 5 we discuss some directions for future research, and state some open problems, defining in the process the concept of isochronous pseudo-almost limit cycles and pseudo-almost isochrons.

2 Preliminary Definitions and Examples

Let the multi-dimensional space $\mathbb{R}^n$ represents all the possible states of a system modeling nonlinear phenomena. The dynamics of the system are determined by the values in $\mathbb{R}^n$ in terms of the time. That is to say we define an evolution map or flow $\Phi$, smooth on the smooth manifold $\mathbb{R}^n$:

$$\Phi : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n,$$

such that $\Phi(x,t) = y$ indicates that the state $x \in \mathbb{R}^n$ evolved into the state $y \in \mathbb{R}^n$ after $t$ units of time, together with the usual flow properties

$$\Phi(x,0) = x, \quad \Phi(x,t_1 + t_2) = \Phi(\Phi(x,t_1), t_2).$$

5 Any pseudo-almost periodic function is also a Besicovitch almost periodic function.
6 The development of the concept of isochrons and the recognition of their significance is due to Winfree (1980).
The flow $\Phi$ then determines a vector field $\mathcal{X}$ (conversely as well) such that, for $x \in M$

$$\mathcal{X}(x) := \frac{\partial \Phi}{\partial t}(x, 0). \quad (2.3)$$

The orbit or trajectory of the flow through $x \in \mathbb{R}^n$ is given by:

$$O(x) := \{ \Phi_x(t) := \Phi(x, t) | t \in \mathbb{R} \}. \quad (2.4)$$

\textbf{Definition 2.1.} The orbit $\gamma = O(x)$ based at $x$ is called a limit cycle if there is a neighborhood $\mathcal{N}$ of $\gamma$ such that $\gamma$ is the only periodic orbit contained in $\mathcal{N}$.

The limit cycle is stable (unstable) if $\omega(s) = \gamma$ ($\alpha(s) = \gamma$) for any $s \in \mathcal{N}$, that is, $\gamma$ is the $\omega$—limit set ($\alpha$—limit set) of any point in $\mathcal{N}$. In other words, the limit cycle, isolated periodic orbit of some period $\tau$, is stable (resp. unstable) if it has a neighborhood $\mathcal{N}$ such that, for some distance function $d$ on $\mathbb{R}^n$, $d(\Phi(y, t), \gamma) \to 0$, as $t \to \infty$ (resp. $t \to -\infty$), for any $y \in \mathcal{N}$.

Note that the phase $\varphi = \frac{t}{\tau}$ of a limit cycle of period $T_0$ refers to the relative position on the orbit, which is measured by the elapsed time (modulo the period) to go from a reference point to the current position on the limit cycle. The most common illustrative examples are from the perturbations of the linear center or linear isochrone.

\section{2.1 Linear center and its perturbations}

\textbf{2.1.1 Poincaré oscillator}

The linear center or \textit{linear isochrone} \footnote{A limit cycle actually controls the behavior of neighboring orbits, attracting/repelling on both sides, or attracting on one side and repelling on the other.} is perturbed into the following system, in polar coordinates $(r, \theta)$

$$\dot{r} = r(1 - r), \quad \dot{\theta} = 1 \quad (2.6)$$

where the origin of the plane is surrounded by a continuum of periodic orbits (not isolated) given by $x^2 + y^2 = c > 0$, is perturbed into the following system, in polar coordinates $(r, \theta)$

$$\dot{r} = r(1 - r), \quad \dot{\theta} = 1 \quad (2.6)$$

The circle $r = 1$ is a $2\pi$—periodic orbit and is unique. It is therefore a limit cycle. Moreover $r$ is a monotone function on each orbit ($\dot{r} > 0$ inside and $< 0$ outside) so that all non constant orbits tend towards the limit cycle which is therefore stable. \footnote{The term isochrone refers to the fact that all the periodic orbits in the continuum have the same constant period normalized to $2\pi$.} \footnote{The Poincaré’s oscillator has been considered a model of biological oscillations, in particular with respect to the effects of periodic stimulation of cardiac oscillators.} \footnote{The isochrons here are radial lines from which the trajectories evolve to equal phase.}
2.1.2 Limit cycles Annulus

The linear center could also be perturbed into a system to generate several limit cycles as in the following example. The $C^\infty$-system
\[
\dot{x} = -y + xp(x, y), \quad \dot{y} = x + yp(x, y),
\]
(2.7)
where
\[
p(x, y) = \sin\left(\frac{1}{x^2 + y^2}\right)e^{-\frac{1}{x^2 + y^2}},
\]
has an infinite number of limit cycles
\[
\gamma_n : \quad x^2 + y^2 = \frac{1}{n\pi}, \quad n \in \mathbb{N}
\]
(2.8)
accumulating at the origin.\[9\]

2.1.3 Remarks

The linear center is a continuum of periodic orbits encircling a critical point. The perturbation in examples 1 and 2 has in fact destroyed these orbits to give birth to respectively a unique limit cycle in example 1, and an accumulating family of limit cycles in example 2. We will see below that a time-dependent pseudo-almost perturbation could lead to the emergence of the so-called pseudo-almost limit cycles.

3 Pseudo-almost limit cycles

3.1 Introductory Concepts

Let $\mathcal{C}(\mathbb{R} \times \Omega, \mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^n$ open, be the Banach space of bounded continuous functions $\phi(t, x)$ endowed with the norm $||\phi|| = \sup_{t \in \mathbb{R}, x \in \Omega} |\phi(t, x)|$. The set $\mathcal{C}(\mathbb{R} \times \Omega, \mathbb{R}^n)$ is a subset of the more general space $L_b(\mathbb{R} \times \Omega, \mathbb{R}^n)$ of all Lebesgue measurable and bounded functions.

Definition 3.1. A function $f$ in $L_b(\mathbb{R} \times \Omega, \mathbb{R}^n)$ is said to be ergodic if for every compact subset $K \subset \Omega$ the mean defined by
\[
\mathcal{M}(f) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(t, x)dt,
\]
exists uniformly for $x \in K$.

We say that the function has a vanishing mean if $\mathcal{M}(f) = 0$. Let $\mathcal{E}(\mathbb{R} \times \Omega, \mathbb{R}^n)$ denote the space of all ergodic functions on $\mathbb{R} \times \Omega$. Note in passing that not all uniformly continuous bounded
functions on $\mathbb{R}$ are ergodic. For instance the function

$$f(t) = \begin{cases} 1 - t^2, & \text{for } |t| < 1, \\ \sin(\log(\frac{1}{t^2})), & \text{for } |t| \geq 1, \end{cases} \tag{3.2}$$

is uniformly continuous in $\mathbb{R}$, but not ergodic.

In the space $L(\mathbb{R} \times \Omega, \mathbb{R}^n)$ of all Lebesgue measurable functions on $\mathbb{R} \times \Omega$, we consider next the following subspace $L_0$ of all functions $\phi: \mathbb{R} \times \Omega \rightarrow \mathbb{R}^n$ such that $\forall x \in \Omega$, $\tilde{\phi}(\cdot) := \phi(\cdot, x)$ is Lebesgue measurable on $\mathbb{R}$ with $M(|\tilde{\phi}|) = 0$, and $M(|\phi|) = 0$.

For example the function

$$\phi(t) = t|\sin \pi t|^N, \quad N > 6, \tag{3.3}$$

is unbounded, Lebesgue measurable with vanishing mean $M$.

The unbounded and discontinuous function

$$\phi(t) := \begin{cases} \sqrt{n}, & n \leq t \leq n + 1/n, \\ 0, & \text{otherwise} \end{cases}, \tag{3.4}$$

is also an element of $L_0$. Indeed we have $\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\phi(t)| dt = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} = 0$.

**Definition 3.2.** The orbit $O(x_0)$ based at $x_0$ as defined above is called a pseudo-almost limit cycle if it is isolated, and more importantly if the function $\Phi(\cdot) := \Phi_{x_0}(\cdot): \mathbb{R} \rightarrow \mathbb{R}^n$ defining the orbit is pseudo-almost periodic in the following sense: $\forall \epsilon > 0$, $\exists \delta = \delta(\epsilon) > 0$, a relatively dense subset $\mathcal{D}_\epsilon \subset \mathbb{R}$, a subset $\mathcal{C}_\epsilon \subset \mathbb{R}$, such that:

1. For $m$ the Lebesgue measure on $\mathbb{R}$,

$$\lim_{t \to \infty} \frac{m(\mathcal{C}_\epsilon \cap [-t, t])}{2t} = 0, \quad (\mathcal{C}_\epsilon \text{ is called an ergodic zero set}), \tag{3.5}$$

2. Let $T_\tau \Phi$ denotes the translate of $\Phi$ by $\tau$, that is, $(T_\tau \Phi(t)) := \Phi(t + \tau)$. Then

$$\|([T_\tau \Phi](t) - \Phi(t))\| < \epsilon, \quad \tau \in \mathcal{D}_\epsilon, \quad t, t + \tau \in \mathbb{R} - \mathcal{C}_\epsilon, \tag{3.6}$$

3. Finally

$$|t_1 - t_2| < \delta \implies \|\Phi(t_1) - \Phi(t_2)\| < \epsilon, \quad t_1, t_2 \in \mathbb{R} - \mathcal{C}_\epsilon. \tag{3.7}$$

Denote $\mathcal{P}_A$ the space of pseudo-almost periodic functions. These functions satisfy the following properties widely available in the relevant literature. $[1] [12] [27]

### 3.1.1 Some properties of pseudo-almost periodicity

We first give an equivalent definition of a pseudo-almost periodic function, in particular in the space $C(\mathbb{R} \times \Omega, \mathbb{R}^n)$, with the restriction of $\mathcal{L}_0$ to the space $\mathcal{E}_0$ containing all functions $\phi \in C(\mathbb{R} \times \Omega, \mathbb{R}^n)$.
such that
\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\phi(t,x)| \, dt = 0,
\] (3.8)
uniformly in \( x \in \Omega \).

**Definition 3.3.** A function \( f : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^n \) is called pseudo-almost periodic in \( t \) uniformly on compact subsets \( K \) of \( \Omega \) if it has a unique decomposition in the form
\[
f(t,x) = a(t,x) + e(t,x),
\] (3.9)
where the component \( a \) is almost periodic, and the component \( e \in E \subset L^0 \) is called the ergodic perturbation of \( f \). Recall that \( a \) is almost periodic if it satisfies the so-called Bohr’s property. That is: \( \forall \epsilon > 0 \exists \tau = \tau(\epsilon) \) such that any interval \( (t,t+\tau) \subset \mathbb{R} \) contains a number \( \tau_\epsilon \), the \( \epsilon \)-almost period or \( \epsilon \)-translation number, such that:
\[
||f(t+\tau,x) - f(t,x)|| < \epsilon, \quad t \in \mathbb{R}, x \in \Omega.
\] (3.10)

We have the following properties relevant to our study and details could be found in Zhang [27] and also in [11, 12].

(1) For \( f \in \mathcal{P}A \), the range \( f(\mathbb{R},K) := \{f(t,x)|t \in \mathbb{R}, x \in K\} \) is bounded for every bounded subset \( K \subset \Omega \).

(2) The function \( f(t,.) \in \mathcal{P}A \) is uniformly continuous in each bounded subset of \( \Omega \) uniformly in \( t \).

(3) When the ergodic zero set \( C_\epsilon = \emptyset \), the space \( \mathcal{P}A \) coincides with the space \( A^P \) of almost periodic functions.

(4) If both functions \( f \) and its derivative \( f' \) are pseudo-almost periodic, with \( f = a + e \) and \( f' = a' + e' \), where \( a \) and \( a' \) in \( \mathcal{P}A \) and \( e \) and \( e' \) in \( L^0 \), then the functions \( a \) and \( e \) are differentiable with \( a' = a \) and \( e' = e \).

(5) The space \( \mathcal{P}A \) is convolution invariant with the space \( L^1(\mathbb{R}) \) of integrable functions on \( \mathbb{R} \).

### 3.1.2 Illustrative Examples

We present some by now classic examples of pseudo-almost periodic functions. See also [12, 27]. We include here their graphics.

(1) **Example 1** The function
\[
\phi_1(t) = \sin t + \sin \sqrt{2}t + \frac{e^{-|t|}}{1 + t^2}
\] (3.11)
has the almost periodic component $a(t) = \sin t + \sin \sqrt{2}t$, and the ergodic perturbation $e(t) = \frac{e^{-|t|}}{1+t^2}$. We represent it along with its components in Figure 1.

Figure 1: $\phi_1(t) = \sin t + \sin \sqrt{2}t + \frac{e^{-|t|}}{1+t^2}$

(2) **Example 2** We have also the function

$$\phi_\omega(t) = I_1(t) + I_2(t), \quad \omega \neq 0,$$

with the almost periodic component

$$I_1(t) = \int_{-\infty}^{\infty} h(t-s)(\sin s + \sin \sqrt{2}s)ds, \quad h \in L^1(\mathbb{R})$$

and the ergodic component

$$I_2(t) = \int_{-\infty}^{\infty} \frac{h(t-s)}{s^2 + \omega^2}ds$$

We take $h(t) = t^2$, in $L^1(\mathbb{R})$, $\omega = 1$ to illustrate in Figure 2.

Figure 2: $\phi_\omega(t) = I_1(t) + I_2(t)$

\[\]
3.2 Existence of Pseudo-almost limit cycles

First note that a periodic or almost periodic function is also pseudo-almost periodic with a zero ergodic perturbation. Consequently a limit cycle is also an almost or a pseudo-almost limit cycle, but not inversely. To make the distinction, we will call strictly pseudo-almost limit cycles those pseudo-almost limit cycles that are not limit cycles.

We start with the case of the linear pseudo-almost center.

3.2.1 Linear pseudo-almost center: an example

Let \( p(t) \in \mathcal{PA}(\mathbb{R}, \mathbb{C}) \) be a complex-valued pseudo-almost periodic function defined on the real numbers, and consider the differential equation (see also [11])

\[
\dot{x}(t) = -\alpha x(t) + p(t), \quad \alpha > 0.
\]  

(3.15)

Define a kernel

\[
K(t) = \begin{cases} 
0, & \text{for } t < 0 \\
\exp(-\alpha t), & \text{for } t \geq 0
\end{cases}
\]

(3.16)

We have \( K \in L^1(\mathbb{R}, \mathbb{C}) \). Thus the convolution

\[
x_\alpha(t) = (K \ast p)(t) = \exp(-\alpha t) \int_{-\infty}^{t} \exp(\alpha s) p(s) \, ds
\]

(3.17)

is also in \( \mathcal{PA}(\mathbb{R}, \mathbb{C}) \), for every \( \alpha > 0 \). Indeed the space \( \mathcal{PA} \) is convolution invariant with \( L^1 \). The equation being linear, it results in the existence of a continuum of parameterized pseudo-almost periodic solutions which we called linear pseudo-almost center. Therefore these solutions are not isolated, and are not pseudo-almost limit cycles.

A graphical representation for the case \( K(t) = t^2, \quad p(t) = \sin t + \sin \sqrt{2}t, \quad \alpha = 1, 2, 3, 4 \) is given in Figure 3.

![Graphical representation](image)

Figure 3: \( x_\alpha(t) = (K \ast p)(t) = \exp(-\alpha t) \int_{-\infty}^{t} \exp(\alpha s) p(s) \, ds. \) \( K(t) = t^2, \quad p(t) = \sin t + \sin \sqrt{2}t. \)
3.2.2 Pseudo-almost periodic perturbations of the harmonic oscillator

Consider the forced oscillations of the harmonic oscillator given by

\[ \ddot{x}(t) + x(t) = f(t) \]

where the forcing term is

\[ f(t) = -\sin \sqrt{2}t + \frac{t^2(t^2 + 4)}{(t^2 + 1)^3} \]

or equivalently, for \( \dot{x} = y \)

\[ \dot{x} = y, \quad \dot{y} = -x + f(t) \] (3.19a)

Clearly the function explicitly given by

\[ x(t) = \sin t + \sin \sqrt{2}t + \frac{1}{t^2 + 1} \] (3.20)

is the unique solution of the equation and it is one of the classic examples of pseudo-almost periodic function that is not periodic. See also [11]. Therefore we obtain an explicit example of pseudo-almost limit cycle.

Figure 4 gives the phase portrait of (3.19a) and the graph of the pseudo-almost periodic function in (3.20).

![Phase portrait and graph of pseudo-almost periodic function](image)

Figure 4 \( \ddot{x}(t) + x(t) = -\sin \sqrt{2}t + \frac{t^2(t^2 + 4)}{(t^2 + 1)^3} \)

We further illustrate the theory of pseudo-almost limit cycles with the well-known Liénard systems.

3.3 Liénard pseudo-almost limit cycles

Liénard equation, which also generalizes the famous Van der Pol oscillator, is ubiquitous in the study of nonlinear systems. Consider the one-parameter family of forced Liénard systems

\[ \ddot{x} + f(x)\dot{x} + g(x) = \mu h(t), \] (3.21)
or equivalently
\[
\begin{align*}
\dot{x} &= y - F(x), \\
\dot{y} &= -g(x) + \mu h(t),
\end{align*}
\] (3.22)
where \(f, g\) are two functions generally nonlinear, continuous and differentiable from \(\mathbb{R}\) to \(\mathbb{R}\), and \(h\) is a time-dependent continuous functions on \(\mathbb{R}\), \(\mu \geq 0\) a small real parameter, and \(F(x) := \int_0^x f(s)ds\).

For the homogeneous Liénard systems at \(\mu = 0\) we recall the following classical result. See more details in, e.g., [5, 10, 18] Theorem 3.4.

**Theorem 3.4.** If the homogeneous Liénard systems satisfy the following conditions:

1. \(f(x)\) is continuous, even and \(f(0) < 0\).
2. \(g(x)\) is locally Lipschitz, odd, and such that \(xg(x) > 0\) for \(x \neq 0\).
3. \(f(x)\) has a unique positive zero at \(x = b\), and it increases at \(\infty\) for \(x > b\).

Then there exists a unique stable limit cycle.

Therefore this theorem provides conditions under which there exists, for the unperturbed Liénard systems, a unique limit cycle, isolated periodic orbit controlling the behavior of neighboring trajectories. We next show that we could subject some classes of Liénard systems to perturbations that, in fact, destroy the limit cycles to give birth to strictly pseudo-almost limit cycles under suitable conditions.

We study system (3.21) or its equivalent form (3.22) under the following additional assumptions:

- **L1:** \(f(x) > 0\), in \(\mathbb{R}\), with \(F(x)\) \(\rightarrow \infty\) as \(|x| \rightarrow \infty\).
- **L2:** \(xg(x) > 0\) for \(x \neq 0\), \(G(x) \rightarrow \infty\) as \(|x| \rightarrow \infty\), with \(G(x) := \int_0^x g(s)ds\).
- **L3:** \(|h(t)| \leq K\), and \(|H(t)| \leq K\), with \(H(t) = \int_0^t h(s)ds\), \(t \in \mathbb{R}\), and \(K\) a positive constant.
- **L4:** \(g'(x) > 0\), and \(g''(x)\) exists and is bounded.

It is known that, under such assumptions, for \(0 < \mu \ll 1\), there exists in the xy-plane a region \(R\) bounded by a regular simple curve (\(C^1\) except possibly at a finite number of points) such that:

1. For every solution \(\gamma(t) = (x(t), y(t))\) of system (3.21) there is a value \(t_0\) such that \(\gamma(t_0) \in R\).
2. If, for a value \(t_0\) of \(t\), we have \(\gamma(t_0) \in R\), then we have also \(\gamma(t) \in R\), for \(t \geq t_0\). That is, solutions entering the set cannot leave it for increasing time.

Moreover the region \(R\) depends only on the functions \(f(x), g(x), h(t)\), the parameter \(\mu\) and the constant \(K\). Equivalently, the region \(R\) may be described by the inequalities \(|x(t)| \leq x_0\) and \(|\dot{x}(t)| \leq v_0\), for a solution \(x(t)\) of the equation (3.22), where \(x_0\) and \(v_0\) are constants independent of \(\mu\). See,
for example, \[7, 15, 21, 23\]. In other words, under the above conditions the solutions ultimately settle in a \(C^1\)-bounded region \(R\) in \(\mathbb{R}^2\). Actually we obtain

**Lemma 3.5.** Assume the conditions \(L_1, \ldots, L_4\). Let \(\gamma(t) = (x(t), y(t))\) be a solution of the system, and \(\tilde{\gamma}(t) = (\tilde{x}(t), \tilde{y}(t))\) either another solution of the system or a solution of an associated system with a sufficiently small perturbation \(\bar{h}(t)\) of the forcing term \(h(t)\). Then we have

\[
\lim_{t \to \infty} |\tilde{\gamma}(t) - \gamma(t)| = 0, \quad (3.23)
\]

Moreover there exists a unique solution \(x(t)\) for all \(t \in \mathbb{R}\).

**Proof.** Let \(\gamma(t) = (x(t), y(t))\) a solution of the system, and \(\tilde{\gamma}(t) = (\tilde{x}(t), \tilde{y}(t))\) either another solution of the system or a solution of an associated system with a sufficiently small perturbation \(\bar{h}(t)\) of the forcing term \(h(t)\).

\[
\lim_{t \to \infty} |\tilde{\gamma}(t) - \gamma(t)| = 0, \quad (3.24)
\]

is equivalent to

\[
\lim_{t \to \infty} |\tilde{x}(t) - x(t)| = 0 = \lim_{t \to \infty} |\tilde{y}(t) - y(t)|. \quad (3.24)
\]

Upon the change of variables \(u(t) = \tilde{x}(t) - x(t), \quad v(t) = \tilde{y}(t) - y(t)\), we obtain the system

\[
\dot{u}(t) = v(t) - \varphi(t)u(t) \\
\dot{v}(t) = -\psi(t)u(t) + \mu \Delta h(t), \quad (3.25)
\]

where

\[
\varphi(t) = \frac{F(x_2) - F(x_1)}{x_2 - x_1}, \quad \psi(t) = \frac{g(x_2) - g(x_1)}{x_2 - x_1}. \quad (3.26)
\]

Note that the function \(f, g'\) and \(g''\) are bounded on the compact region \(R\). For sufficiently small values of the parameter \(\mu \ll 1\), we can construct a Lyapunov-type quadratic form

\[
V(t, u, v) = \psi(t)u^2 + v^2 - 2cvu, \quad (3.27)
\]

with \(c > 0\) chosen small enough for \(V(t, u, v)\) to be positive definite such that

\[
V(t, u, v) \geq c(u^2 + v^2), \quad (3.28)
\]

\(c\) a positive constant, and such that

\[
\dot{V}(t, u, v) + cV(t, u, v) < 0. \quad (3.29)
\]

Actually we have

\[
\frac{dV}{dt}(t, u, v) = -2(\varphi \psi - \dot{\psi} - 2c\psi)u^2 - 2cv^2 + 2c\varphi uv, \quad (3.30)
\]

yielding

\[
\dot{V}(t, u, v) := \dot{V}(t, u, v) + cV(t, u, v) = -(2\varphi \psi - \dot{\psi} - 3c\psi)u^2 + 2c(\varphi - c)uv - cv^2. \quad (3.31)
\]
The quadratic form $\tilde{V}(t,u,v)$ can be made negative definite by taking the constant $c$ such that

$$c < \frac{2\varphi\psi - \dot{\psi}}{3\varphi}, \quad c(3\varphi + (\varphi - c)^2) < 2\varphi\psi - \dot{\psi}, \quad (3.32)$$

which entails

$$\dot{V}(t,u,v) < V(t_0) e^{-c|t-t_0|}. \quad (3.33)$$

Therefore $V(t) \to 0$ as $t \to \infty$, implying that $u \to 0$ and $v \to 0$. The constant $c$ is appropriately chosen so that, when $|\Delta h(t)| = |\tilde{h}(t) - h(t)| \to 0$, we can make $V(t) \to 0$ for $t \to \infty$. That is, the solutions of the system of the perturbed forcing term ultimately converge to the solutions of the original system.

Next let $\gamma(t) = (x(t),y(t))$ be one of these solutions which settled in $R$ for $t \geq t_0$. We then define the sequence of solutions $\gamma_n(t) = \gamma(t+n) = (x_n(t),y_n(t)), \quad t \geq t_0 - n$. The sequence is therefore equicontinuous and uniformly bounded. Consequently we can extract a subsequence $\gamma_{nk}(t)$ converging uniformly to a solution $\bar{\gamma}(t) = (\bar{x}(t),\bar{y}(t))$ lying completely in $R$ for all $t \in R$. (\lim_{n \to \infty}(t_0 + n, \infty) = (-\infty, \infty)).$ And of course $\bar{\gamma}(t)$ is unique.

3.3.1 Remarks

Indeed the proof of the theorem actually accomplishes the followings: the solutions of the system associated to the perturbed forcing term ultimately converge to the solutions of the original system; moreover, under the assumptions above, only one solution of the system settles in the bounded region $R$ for all time in $R$; as we show below, that single solution will be of the same nature as the forcing term, when it becomes pseudo-almost periodic.

In a previous work, [24] the case of the pseudo-almost periodic forcing was presented as a corollary to that of almost periodic forcing; here we present a more elegant and self-contained proof drawing from the above definitions of pseudo-almost periodicity, definitions not used in the cited work.

We state and prove

**Theorem 3.6.** Assume the forcing term $h(t)$ is a pseudo-almost periodic function. Then under the conditions $L_1, \ldots, L_4$, the forced Liénard system exhibits a unique asymptotically stable pseudo-almost limit cycle.

**Proof.** The proof is based on the previous lemma, including the existence of a unique solution enclosed in $R$ for all time. First assuming the forcing term $h(t)$ is pseudo-almost periodic entails from the definition above that, for any arbitrary $\varepsilon$, there exists $\delta = \delta(\varepsilon)$, an $\varepsilon$–pseudo-almost period $\tau \in D_\varepsilon$, a relatively dense set in $R$ such that

$$||h(t+\tau) - h(t)|| < \varepsilon, \quad t, t + \tau \in R - C_\varepsilon \quad (3.34)$$
and
\[ |t_1 - t_2| < \delta \implies \|h(t_1) - h(t_2)\| < \epsilon, \quad t_1, t_2 \in \mathbb{R} - C_\epsilon, \]  
(3.35)
where \(C_\epsilon\) is the ergodic zero set defined above. For such an \(\epsilon\)-pseudo-almost period consider the unique solution \(\tilde{\gamma}(t)\) given in the previous lemma that settles in \(\mathbb{R}\) for all time \(t \in (-\infty, \infty)\), and the associated function \(\tilde{\gamma}(t + \tau) = (\tilde{x}(t + \tau), \tilde{y}(t + \tau))\). This function is readily a solution of the following system (\(\mathcal{E}_\epsilon\))
\[ \dot{x} = y - F(x), \quad \dot{y} = -g(x) + \mu h(t + \tau), \]  
(3.36)
Take \(h(t + \tau)\) as a sufficiently small perturbation of \(h(t)\) as above. Therefore, according to the previous propositions, the solutions \(\tilde{\gamma}(t)\) and \(\tilde{\gamma}(t + \tau)\) converge. Thus we obtain
\[ \|\tilde{\gamma}(t + \tau) - \tilde{\gamma}(t)\| < \epsilon, \quad t, t + \tau \in \mathbb{R} - C_\epsilon. \]  
(3.37)
Moreover we also have, for \(t_1, t_2 \in \mathbb{R} - C_\epsilon\),
\[ |\tilde{\gamma}(t_2) - \tilde{\gamma}(t_1)| \leq |t_2 - t_1| \sup_{\mathbb{R}}|\dot{y}|, \]  
(3.38)
which ensures the existence of \(\delta\) such that
\[ |t_1 - t_2| < \delta \implies \|\tilde{\gamma}(t_1) - \tilde{\gamma}(t_2)\| < \epsilon, \quad t_1, t_2 \in \mathbb{R} - C_\epsilon, \]  
(3.39)
Therefore we conclude that the unique solution \(\tilde{\gamma}(t)\) is pseudo-almost periodic.

Moreover, from the previous lemma, all other solutions of the system that ultimately settle in \(\mathbb{R}\) converge to this unique pseudo-almost periodic solution \(\tilde{\gamma}(t) \in \mathbb{R}\). Therefore the system has a unique (isolated) almost periodic solution to which any other solution unwinds in the \(C^1\)-bounded set \(\mathbb{R}\). It is a stable pseudo-almost limit cycle as defined above. Hence the claim. \(\square\)

4 Pseudo-almost periodic Waves

The importance of Liénard systems among nonlinear systems also comes from the fact that several systems can be transformed into Liénard systems and solved. \([1, 19]\). We present next some partial differential equations solvable first by reducing them to some Liénard-type equations, then by applying the previous theorems.

4.1 Hyperbolic pseudo-almost periodic Wave

Consider systems described by the time-perturbed nonlinear hyperbolic equation
\[ u_{tt} = u_{xx} + f_0(u)u_x + g_0(u) + p(t) \quad (\mathcal{H}) \]
The search of special solutions of the form
\[ u(x, t) = y(x + \lambda t), \quad \lambda \in \mathbb{R} \]  
(4.1)

defining the wave with speed \( v = |\lambda| \), yields the Liénard-type equation
\[ (1 - \lambda^2)\ddot{y} + f_0(y)\dot{y} + g_0(y) = -p(t) \]  
(4.2)

Define \( f(y) = \frac{f_0(y)}{1 - \lambda^2}, \) \( g(y) = \frac{g_0(y)}{1 - \lambda^2}, \) and \( h(t) = \frac{-p(t)}{1 - \lambda^2}. \) The functions \( f_0 \) and \( g_0 \) are continuously differentiable chosen together with the speed \( v = |\lambda| \) of the waves \( u(t, x) \) such that the function \( f, g, \) and \( h \) satisfy the assumptions \( L_1, \ldots, L_4. \) Obviously assuming \( p(t) \) pseudo-almost periodic implies \( h(t) \) is pseudo-almost periodic. Therefore we conclude under these assumptions

**Theorem 4.1.** For a pseudo-almost periodic perturbation \( p(t), \) the nonlinear hyperbolic equation \( (H) \) has a pseudo-almost periodic solitary wave \( u(x, t) = y(x + \lambda t), \) where \( y(x) \) is the unique pseudo-almost limit cycle of the perturbed Liénard-type equation \( (4.2). \)

**Proof.** The proof is immediate and is adapted from theorem (3.6). \( \square \)

We next consider a parabolic partial differential equation describing a reaction-diffusion model.

### 4.2 Parabolic pseudo-almost periodic Wave: a reaction-diffusion model

Consider now the time-perturbed parabolic equation describing a reaction-diffusion model
\[ u_t = u_{xx} + f_0(u)u_x + g_0(u) + p(t) \]  
(\( RD \))

Looking again for special solutions of the form \( (4.1) \) leads to the Liénard-type equation
\[ \ddot{y} + f_0(y) - \lambda \dot{y} + g_0(y) = 0 \]  
(4.3)

As in the previous case we set \( f(y) = f_0(y) - \lambda, \) \( g(y) = g_0(y), \) and \( h(t) = -p(t). \) The functions \( f_0 \) and \( g_0 \) are continuously differentiable and determined together with the speed \( |\lambda| \) of the waves \( u(t, x) \) such that the function \( f, g, \) and \( h \) satisfy the assumptions \( L_1, \ldots, L_4. \) Again assuming \( p(t) \) pseudo-almost periodic implies \( h(t) \) is also pseudo-almost periodic. We therefore obtain the equivalent theorems of existence of pseudo-almost solitary waves to the reaction-diffusion equation as functions of the corresponding Liénard pseudo-almost limit cycles. That is,

**Theorem 4.2.** For a pseudo-almost periodic perturbation \( p(t), \) the nonlinear parabolic equation \( (RD) \) has a pseudo-almost periodic solitary wave \( u(x, t) = y(x + \lambda t), \) where \( y(x) \) is the unique pseudo-almost limit cycle of the perturbed Liénard-type equation \( (4.3). \)
5 Outlook and Open Problems

Arnold in [3] states

Une trajectoire fermée non dégénérée ne disparaît pas par une petite déformation du système, mais se déforme légèrement. Donc le système des trajectoires est structurellement stable dans le voisinage de la trajectoire fermée générique.

That is, periodic orbits do not just disappear under small perturbation, but they may be slightly deformed, due to the fact that the system of trajectories is structurally stable in the neighborhood of a periodic orbit. Many forced systems such as the Liénard ones are actually small perturbations of systems having periodic orbits (limit cycles) in their unperturbed form, and many results do imply the disappearance of these orbits upon perturbation. The appearing of pseudo-almost periodic solutions could result from the deformation/bifurcation of existing orbits. Therefore one must investigate the relation between the “new” pseudo-almost periodic solutions appearing upon perturbation and the periodic-type orbits of the unperturbed system, including the question in the following Open Problem 1.

(1) Open Problem 1: Co-existence of limit cycles and strictly pseudo-almost limit cycles

For parameterized systems, including the above Liénard systems, investigate conditions under which co-exist limit cycles and strictly almost or pseudo-almost limit cycles partitioning the phase space.

(2) Open Problem 2: Isochronous pseudo-almost limit cycles

Let $\gamma$ be a strictly pseudo-almost limit cycle of a flow $\phi$ on $\mathbb{R}^n$. A point $x_1$ in $\mathbb{R}^n$ has asymptotic phase with respect to $\gamma$ if there is a point $x_0 \in \gamma$ such that $\lim_{t \to \pm \infty} |\phi_t(x_1) - \phi_t(x_0)| = 0$. We say that $x_1$ is in phase with $x_0$.

It is well known that a hyperbolic limit cycle has some neighborhood where every point has asymptotic phase with respect to the limit cycle, due to the existence of invariant foliation [8]. Similar question needs to be addressed as well in case of strictly pseudo-almost limit cycles.

**Definition 5.1.** A strictly pseudo-almost limit cycle is said to be isochronous if there is a neighborhood of $\gamma$ in which every point is in phase with a point on $\gamma$.

In the case of limit cycles, we have, for instance, the following examples. System

$$\dot{r} = -\frac{1}{3}(r-1)^4 e^{-|r-1|^{-3}}, \quad \dot{\theta} = 2\pi$$

has a nonhyperbolic limit cycle at the unit cycle with period 1, attracting for $r > 1$. The asymptotic phase of any point $(r_0, \theta_0)$ in its neighborhood is $(1, \theta_0)$. The limit cycle is therefore isochronous. For more details see [8].

It would be interesting to:
(a) Perturb system (5.1), in particular in the angle variable, and study the conditions of appearance of strictly pseudo-almost limit cycles.

(b) Investigate the conditions of existence of isochronous strictly pseudo-almost limit cycles, in particular for the forced Liénard systems.

(c) Investigate the bifurcation of pseudo-almost limit cycles from an isochronous period annulus, as in [25].

(3) Open Problem 3: **Pseudo-almost isochrons**

As above, we further define:

**Definition 5.2.** Given $x_0 \in \gamma$ where $\gamma$ is a strictly pseudo-almost limit cycle, a pseudo-almost isochron $I(x_0)$ based at $x_0$ is the set of all point $x \in \mathbb{R}^n$ in phase with $x_0$.

As in the case of limit cycles we conjecture the existence of pseudo-almost isochrons, and that they will foliate the neighborhood of pseudo-almost limit cycles. Their determination is definitely an interesting but difficult question of research. One line of attack might be similar to Guckenheimer and Winfree investigation of isochrons of limit cycles. [16, 20]

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On the Poisson’s equation $-\Delta u = \infty$.

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ABSTRACT

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. We proof the existence of a bounded solution of the Poisson’s equation $-\Delta u = \infty$ on $\Omega$.

RESUMEN

Sea $\Omega \subset \mathbb{R}^N$ un dominio acotado. Probamos la existencia de una solución acotada para la ecuación de Poisson $-\Delta u = \infty$ en $\Omega$.

Keywords and Phrases: Newtonian potential; nonlinear analysis; celestial mechanics

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\footnote{Dedicated to Professor Gaston M N’Guérékata on the occasion of his 60\textsuperscript{th} birthday.}
1 Introduction.

In [19] it is stated that

Le mouvement d’un corps libre consiste dans le mouvement de translation de son centre de gravité et dans le changement de sa position autour de ce point. La recherche du mouvement du centre de gravité se réduit à déterminer le mouvement d’un point sollicité par des forces données; et, relativement aux corps célestes, ces forces sont le résultat des attractions de sphéroïdes dont la figure est supposée connu. Soient \( dm \) une molécule d’un sphéroïde; \( x', y', z' \) les trois coordonnées orthogonales de cette molécule; \( dm \) sera de la forme \( \xi dx' dy' dz' \), \( \xi \) étant fonction de \( x', y', z' \). Soient encore \( x, y, z \) les coordonnées d’un point attiré, on aura

\[
V = \int G \frac{\xi dx' dy' dz'}{\sqrt{(x' - x)^2 + (y' - y)^2 + (z' - y)^2}}
\]

(1)

dest cette intégrale étant prise relativement à toute l’étendue du sphéroïde. Ses limites étant indépendantes de \( x, y, z \) ainsi que les variables \( x', y', z' \), il est clair qu’en differential l’expression de \( V \) par rapport \( ax, y, z \) il suffira, dans cette différentiation, d’avoir égard au radical que renferme cette expression, et alors il est facile de voir que l’on aura

\[
0 = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial y^2}.
\]

(2)

In modern interpretation of potential \( V \) of mass distributions, we have

\[
V(x, y, z) = \int_G \frac{\xi(x', y', z') dx' dy' dz'}{\sqrt{(x' - x)^2 + (y' - y)^2 + (z' - y)^2}}.
\]

(3)

where \( \xi(x', y', z') \) is the density of a mass distribution in the space \( x', y', z' \). Then \( \nabla V \) furnishes the gravity field force and \( -\Delta V = 0 \) on \( \mathbb{R}^3 \setminus G \).

In 1813 Poisson found that for a ball \( G \) the following equation is valid in the case of constant density \( \xi(x, y, z') = \rho \)

\[
-\Delta u = 4\pi\rho \text{ on } G \text{ Poisson’s equation.}
\]

Therefore a natural question is: there exists a solution for Poisson’s equation with \( \rho = \infty \)? That kind of solution will be related to gravity potential of bodies with infinite density or black holes. The authors are not aware of a previous result deducing the existence of black holes using Newton gravity theory or the gravity potential inside of a black hole. The equation

\[
-\Delta u = u^p,
\]

(4)

for \( p \) a nonnegative real number and \( u > 0 \)in a Ball of radius \( R \) in \( \mathbb{R}^3 \), with Dirichlet boundary conditions was introduced by Lane [18] for modelling both the temperature and the density of
mass on the surface of the sun. Today the problem \([1]\) is named Lane-Emden-Fowler equation. It was used first in the mid-19th century in the study of internal structure of stars mainly by Chandrasekhar \([4, 7, 9]\). Singular Lane-Emden-Fowler equations \((p < 0)\) has been considered in a remarkable pioneering paper by Fulks and Maybe \([10]\).

Eddington \([6]\) proposed the equation
\[
-\Delta u = \exp(2u) \frac{1}{1 + |x|^2} \text{ in } \mathbb{R}^3,
\] (5)
in order to represent the gravitational potential \(u\) of a globular cluster of stars.

Matukuma \([20]\) introduced the equation
\[
-\Delta u = \frac{u^r}{1 + |x|^2} \text{ in } \mathbb{R}^3,
\] (6)
where \(u\) is the gravity potential, \(\rho = (2\pi)^{-1} (1 + |x|^2)^{-1} u^r\) is the density and \(\int_{\mathbb{R}^3} \rho \text{d}x\) is the total mass to study the gravitational potential \(u\) of a globular cluster of stars. For the same problem Hénon \([15]\) suggested
\[
-\Delta u = |x|^l u^r \text{ in } \Omega \subset \mathbb{R}^3.
\] (7)
Black holes solutions means that the gravitational potential of the cluster behaves like \(\frac{1}{r}\) \((r = |x|)\) near the center.

Peebles \([16, 17]\) gives for the first time a derivation of the steady state distribution of the star near a massive collapsed object. The question of the existence of black hole in a globular cluster is still open (1995). Core collapse does occur, for instance using Hubble Space Telescope, Bendinelli et.al. \([2]\) reported the first detection of a collapsed core globular cluster in M31.

On May 25, 1994 astronomers at NASA headquarters announced the Hubble Space Telescope finding of a supermassive black hole in the heart of the giant galaxy M87, more than 50 million light-years.

The equation
\[
-\Delta \frac{1}{|x - x_0|} = 4\pi \delta(x - x_0) \text{ in } \mathbb{R}^3,
\]
has a deep insight because relate the formulation of the Laplace operator and the Dirac \(\delta\) function in a weak sense. The Laplace operator with point interaction in \(\mathbb{R}^3\) given by \(-\Delta + \alpha \delta, \alpha \in \mathbb{R}\) has been widely study for your applications in quantum physics (see for example \([11]\)) and in seismic imaging \([3]\).

Our purpose in this paper is to give a classical interpretation to the equation
\[
-\Delta u = \infty \text{ in } \Omega \subset \mathbb{R}^N.
\] (8)
We define:
Definition 1.1. The equation (8) has a classical solution if there exist two non decreasing sequences of functions \( \{u_j\}_{j=1}^{\infty} \in C(\Omega) \cap C^2(\Omega) \) and \( \{f_j\}_{j=1}^{\infty} \) such that
\[
-\Delta u_j = f_j \quad \text{in} \ \Omega,
\]
and \( \lim_{j \to \infty} f_j(x) = \infty \) for all \( x \in \Omega \) and \( \lim_{j \to \infty} u_j(x) = u(x) < \infty \) for all \( x \in \Omega \).

Our main result in this article is as follows.

Theorem 1.2. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \), \( N \geq 3 \). Then the problem
\[
-\Delta u = \infty \quad \text{in} \ \Omega,
\]
has a non negative classical solution \( u \).

Under the authors knowledge this is the first compactness result dealing with infinite on a non trivial domain (see for example [21] first chapter: direct methods in the calculus of variations). Similarly the theory of generalized functions not allow solutions to this kind of problem because every distribution is locally a Newtonian potential:

Theorem 1.3. (page 277 [2]) Let \( \Omega \) be an open set of \( \mathbb{R}^N \), \( f \in \mathcal{D}'(\Omega) \) and \( u \) a solution (in the sense of distributions) of Poisson’s equation \( \Delta u = f \) on \( \Omega \). Then for every bounded open set \( \Omega_1 \) with \( \overline{\Omega_1} \subset \Omega \) there exists \( f_1 \in \mathcal{E}' \) the space of distributions on \( \mathbb{R}^N \) with compact support, such that \( f_1 = f \) on \( \Omega \) and \( u = \) the Newtonian potential of \( f_1 \) on \( \Omega_1 \).

Moreover if we study this problem using a weak formulation in Sobolev’s spaces, the Georgi-Nash-Moser theory cannot be used to derive any comparable compactness result [14].

We will use a non linear singular elliptic approach as in [1, 8, 13] to obtain the result.

Our strategy is study the auxiliary problem
\[
-\Delta u_{e,m} = g_m(u_e) \quad \text{in} \ \Omega,
\]
\[
u_{e,m} = e \quad \text{on} \ \partial \Omega,
\]
where \( g_m : (0, \infty) \to (0, \infty) \), \( m = 1, \ldots, \infty \) is non increasing locally Hölder continuous function singular at the origin with the properties \( g_m(s) = g(s) \) for all \( s \geq 1 \) and \( \lim_{m \to \infty} g_m(s) = \infty \) for all \( s \in (0, 1) \), \( m = 1, \ldots, \infty \) and \( g : (0, \infty) \to (0, \infty) \) is strictly non increasing locally Hölder continuous function singular at the origin.

Our result [12] is obtained letting \( \lim_{m \to \infty, e \to 0^+} u_{e,m} \). This limit by definition has not weak derivatives of first or second order.
2 Auxiliary results

**Theorem 2.1** ([1]). Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, $g : (0, \infty) \to (0, \infty)$ is non increasing locally Hölder continuous function (that may be singular at the origin). Then the problem

$$-\Delta u = g(u) \quad \text{in } \Omega,$$

$$u = \varepsilon \text{ on } \partial \Omega,$$

has a unique positive solution $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ for $\varepsilon \geq 0$. Moreover $u_{\varepsilon_2} \geq u_{\varepsilon_1}$ for $\varepsilon_2 \geq \varepsilon_1$.

We consider the auxiliary problem

$$-\Delta u_m = g_m(u_m) \quad \text{in } \Omega,$$

$$u_m = 0 \text{ on } \partial \Omega,$$

(10)

**Lemma 2.2.** Let $u_m$ be a solution of the equation (10). Then $u_{m+1} \geq u_m$.

**Proof.** Suppose that there exists $x_0 \in \Omega$ such that $u_m(x_0) > u_{m+1}(x_0)$. Therefore for $\tau > 0$ small enough we have the inequality $u_m(x_0) > \tau + u_{m+1}(x_0)$. Then by continuity in $\Omega$ of the function $F(x) = u_m(x) - \tau - u_{m+1}(x)$ there exist a non empty open set $\Omega_\tau$ such that $F(x) > 0$ for all $x \in \Omega_\tau$ and $F = 0$ on $\partial \Omega_\tau$. Using that $u_m(x) > \tau + u_{m+1}(x)$ for all $x \in \Omega_\tau$, we deduce

$$g_m(u_m(x)) \leq g_{m+1}(u_m(x)) \leq (\tau + u_{m+1}(x)) \leq g_{m+1}(u_{m+1}(x))$$

for all $x \in \Omega_\tau$. Then

$$-\Delta u_m \leq -\Delta (u_{m+1} + \tau) \quad \text{in } \Omega_\tau,$$

$$u_m = u_{m+1} + \tau \quad \text{on } \partial \Omega_\tau,$$

and we obtain $u_m \leq u_{m+1} + \tau$ in $\Omega_\tau$ (Theorem 3.3 [13]) a contradiction.

**Lemma 2.3.** Let $u_m$ be a solution of the equation (10). Then $g_{m+1}(u_{m+1}(x)) \geq g_m(u_m(x))$.

**Proof.** Suppose that there exists $x_0 \in \Omega$ such that $g_m(u_m(x_0)) > g_{m+1}(u_{m+1}(x_0))$. Then by continuity in $\Omega$ of the function $H(x) = g_m(u_m(x)) - g_{m+1}(u_{m+1}(x))$, there exists $\Omega \subset \Omega$ such that $H(x) > 0$ in $\hat{\Omega}$ and $H(x) = 0$ on $\partial \hat{\Omega}$

$$-\Delta u_m \geq -\Delta u_{m+1} \quad \text{in } \hat{\Omega},$$

$$u_m = u_{m+1} \quad \text{on } \partial \hat{\Omega}.$$ We imply $u_m \geq u_{m+1}$ in $\hat{\Omega}$ (Theorem 3.3 [13]). Therefore $g_m(u_m(x)) \leq g_m(u_{m+1}(x)) \leq g_{m+1}(u_{m+1}(x))$ for all $x \in \hat{\Omega}$. A contradiction.

**Remark 2.4.** In the proof of Lemmas 2.2 and 2.3 it is assumed only that $g_m$ is a non increasing continuous function.
3 Proof

Proof of Theorem 1.2. Let us consider the problem

\[-\Delta v = g(v) \quad \text{in} \ \Omega,\]
\[v = 0 \quad \text{on} \ \partial\Omega.\]

We introduce the equations

\[-\Delta e = g(e) \quad \text{in} \ \Omega,\]
\[e = 1 \quad \text{on} \ \partial\Omega.\]

\[-\Delta w = g(e) \quad \text{in} \ \Omega,\]
\[w = 0 \quad \text{on} \ \partial\Omega.\]

Using \(v \leq e\) (see Lemma 2.3 and 2.6 in [1]), we infer

\[-\Delta w = g(e) \leq g(v) = -\Delta v \quad \text{in} \ \Omega,\]
\[w = 0 = v \quad \text{on} \ \partial\Omega.\]

Then \(w \leq v\) in \(\Omega\). Setting \(g_0 = g\) and using the auxiliary results with the new sequence \(\{g_j\}_{j=0}^{\infty}\), we conclude that \(w \leq u_m \leq e\) for \(m = 1, \ldots, \infty\). Using Lemma 2.2, we infer the existence of \(\lim_{m \to \infty} u_m(x) = u(x)\) for all \(x \in \Omega\). We restrict ourselves to the situation \(\Omega = B_1(0)\) where \(B_1(0)\) is the ball of radius 1 with center at 0. Applying the main result of [12], we infer that \(u_m\) is a radial function with \(\frac{\partial u_m}{\partial r} < 0\). Therefore \(u\) is also a radial non increasing function.

We proceed by contradiction, suppose that

\[\lim_{m \to \infty} g_m(u_m(x)) < \infty \quad \text{for all} \quad 0 \leq \|x\| < 1.\]

Our first implication is that the function \(u\) is strictly non increasing, because if exists \((r_1, r_2)\) with \(r_2 < 1\), and \(u(r_1) = u(r_2)\).

Then \(-\Delta u = 0\) on the annulus \(A(r_1, r_2)\). Using Theorem 9.11 page 235 in [14], we deduce

\[\|u_m\|_{H^{1,p}(\Omega')} \leq C(N, p, \Omega', A(r_1, r_2)) \left( \|u_m\|_{L^p(A(r_1, r_2))} + \|g(u_m)\|_{L^p(A(r_1, r_2))} \right)\]
\[\leq C(N, p, \Omega', A(r_1, r_2)) \left( \|e\|_{L^p(A(r_1, r_2))} + \limsup_{m \to \infty} g_m(u_m(r_2)) \right),\]

for all \(p > N\), therefore by Sobolev’s embedding theorem (Theo. 7.26 [14]) we deduce \(\|u_m\|_{C^{1, \alpha}(\Omega')} \leq C\). We use a non negative test function \(\varphi\) with support contained in \(\Omega'\):

\[0 = \int_{\Omega'} \nabla u \cdot \nabla \varphi dx = \lim_{m \to \infty} \int_{\Omega'} \nabla u_m \cdot \nabla \varphi dx\]
\[= \int_{\Omega'} g_m(u_m) \varphi dx \geq \int_{\Omega'} g_0(u_0) \varphi dx > 0.\]
Contradiction, therefore we deduce that \( u \) is a strictly non increasing function. Moreover using again estimates in Theorems 9.11 and 9.12 in [14] we have \( u \in C^1_{\text{loc}}(B_1(0)) \).

By assumption \( \limsup_{r \to 1} u(r) \geq 1 \), therefore \( u(r) > 1 \) for \( 0 \leq r < 1 \). By construction there exists \( 0 < r_0 < 1 \) such that \( g_0(u(r_0)) > g_0(1) \).

Using Lemma 2.3 we derive \( g_0(u(r_0)) \leq g_m(u_m(r_0)) \). But \( \lim_{m \to \infty} u_m(r_0) = u(r_0) > 1 \) and therefore for \( m \) big enough \( u_m(r_0) > 1 \). Moreover \( g_m(u_m(r_0)) = g_0(u_m(r_0)) < g_0(1) \) because \( g_0 \) is strictly non increasing.

Contradiction. It is follows that there exists \( 0 \leq r_1 < 1 \) such that

\[
\lim_{m \to \infty} g_m(u_m(r_1)) = \infty.
\]

Now, because \( u_m \) is a radial non increasing function, we infer that

\[
g_m(u_m(r_1)) \leq g_m(u_m(r))
\]

for all \( r_1 < r < 1 \). So

\[
\lim_{m \to \infty} g_m(u_m(r)) = \infty \quad \text{for all} \quad r_1 \leq r < 1.
\]

Now for \( \Omega \) a bounded domain in \( \mathbb{R}^N \), \( N \geq 3 \) consider the transformation \( u_m(\frac{a+x}{r}) \). This end the proof.

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References


Generalized Ulam - Hyers Stability of Derivations of a AQ - Functional Equation

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ABSTRACT

In this paper, the author established the generalized Ulam - Hyers stability of Derivations of additive and quadratic (AQ)- functional equation

\[ f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y). \]

RESUMEN

En este artículo el autor establece la estabilidad generalizada Ulam-Hyers de derivaciones de la ecuación (AQ)-funcional cuadrática y aditiva

\[ f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y). \]

Keywords and Phrases: Additive functional equations, Quadratic functional equation, Mixed type functional equation, Additive derivations, Quadratic derivations, Ulam - Hyers stability

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1 Introduction

The study of stability problems for functional equations is related to a question of Ulam [33] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [13]. It was further generalized and excellent results obtained by number of authors [2, 10, 24, 30, 32].

Over the last six or seven decades, the above problem was tackled by numerous authors and its solutions via various forms of functional equations like additive, quadratic, cubic, quartic, mixed type functional equations which involves only these types of functional equations were discussed. We refer the interested readers for more information on such problems to the monographs [1, 9, 14, 19, 22, 31].

C. Park [25] applied Gavruta’s result to Banach modules over a $C^*$—algebra. Many authors have studied the structure of $C^*$—algebras for different types of functional equations in various settings one can refer [6, 8, 20, 28]. It seems that approximate derivations was first investigated by K.W. Jun and D.W. Park [16]. Recently, the stability of derivations have been investigated in [7, 11, 21, 27, 29] and references therein. The stability of cubic derivations was first time introduced and investigated by M.E. Gordji et. al. [12]. With the help of [12], the stability of quadratic derivations was discussed by M. Arunkumar et. al. [3].

Very recently M. Arunkumar and J.M. Rassias [5], established the generalized Ulam - Hyers stability of an additive and quadratic (AQ)-mixed type functional equation
\[
f(x + y) + f(x - y) = 2f(x) + f(y) + f(-y)
\]
in Banach spaces. The solution and stability of several types of Mixed type additive and quadratic type functional equations were discussed in [4, 17, 18, 21, 23].

In this paper, the author first time established the generalized Ulam - Hyers stability of mixed derivations of a additive - quadratic (AQ)-functional equation (1).

Hereafter through out this paper, let us consider $X$ and $Y$ to be a normed Algebra and a Banach Algebra, respectively.

2 Stability Results: Additive Derivations

In this section, the authors investigate the generalized Ulam-Hyers stability of additive derivations of the AQ-functional equation (1).

Definition 2.1. A $C$—linear mapping $A : X \to X$ is called Additive Derivation on $X$ if $A$ satisfies
\[
A(xy) = A(x)y + xA(y)
\]
for all $x, y \in X$.
Theorem 2.1. Let \( j = \pm 1 \). Let \( f_a : X \to Y \) be an odd mapping for which there exist a function \( \alpha, \beta : X^2 \to [0, \infty) \) with the condition

\[
\sum_{n=0}^{\infty} \alpha \left( \frac{2^{n}x, 2^{n}y}{2^{n}} \right) \text{ converges in } \mathbb{R} \quad \text{and} \quad \lim_{n \to \infty} \alpha \left( \frac{2^{n}x, 2^{n}y}{2^{n}} \right) = 0
\]

\[
\sum_{n=0}^{\infty} \beta \left( \frac{2^{n}x, 2^{n}y}{2^{2n}} \right) \text{ converges in } \mathbb{R} \quad \text{and} \quad \lim_{n \to \infty} \alpha \left( \frac{2^{n}x, 2^{n}y}{2^{2n}} \right) = 0
\]

such that the functional inequalities

\[
\|f_a(x + y) + f_a(x - y) - 2f_a(x) - f_a(y) - f_a(-y)\| \leq \alpha(x, y)
\]

and

\[
\|f_a(xy) - f_a(x)y - xf_a(y)\| \leq \beta(x, y)
\]

for all \( x, y \in X \). Then there exists a unique Additive Derivation mapping \( A : X \to Y \) satisfying the functional equation (1) and

\[
\|f_a(x) - A(x)\| \leq \frac{1}{2} \sum_{k=1}^{\infty} \frac{\alpha(2^{k}x, 2^{k}x)}{2^{k^2}}
\]

for all \( x \in X \). The mapping \( A(x) \) is defined by

\[
A(x) = \lim_{n \to \infty} \frac{f_a(2^{n}x)}{2^{n}}
\]

for all \( x \in X \).

Proof. Assume \( j = 1 \). Replacing \( y \) by \( x \) in (1) and using oddness of \( f \), we get

\[
\left\| f_a(x) - \frac{f_a(2x)}{2} \right\| \leq \frac{\alpha(x, x)}{2}
\]

for all \( x \in X \). Now replacing \( x \) by \( 2x \) and dividing by 2 in (8), we get

\[
\left\| \frac{f_a(2x)}{2} - \frac{f_a(2^2x)}{2^2} \right\| \leq \frac{\alpha(2x, 2x)}{2^2}
\]

for all \( x \in X \). From (8) and (9), we obtain

\[
\left\| f_a(x) - \frac{f_a(2^2x)}{2^2} \right\| \leq \left\| f_a(x) - \frac{f_a(2x)}{2} \right\| + \left\| \frac{f_a(2x)}{2} - \frac{f_a(2^2x)}{2^2} \right\| \\
\leq \frac{1}{2} \left[ \alpha(x, x) + \frac{\alpha(2x, 2x)}{2} \right]
\]

(10)
for all \( x \in X \). In general for any positive integer \( n \), we get
\[
\left\| f_a(x) - \frac{f_a(2^n x)}{2^n} \right\| \leq \frac{1}{2} \sum_{k=0}^{n-1} \frac{\alpha(2^k x, 2^k x)}{2^k}
\]
(11)
\[
\leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\alpha(2^k x, 2^k x)}{2^k}
\]
for all \( x \in X \). In order to prove the convergence of the sequence
\[
\left\{ \frac{f_a(2^n x)}{2^n} \right\}
\]
replace \( x \) by \( 2^m x \) and dividing by \( 2^m \) in (11), for any \( m, n > 0 \), we deduce
\[
\left\| f_a(2^m x) - \frac{f_a(2^{n+m} x)}{2^{n+m}} \right\| = \frac{1}{2^m} \left\| f_a(2^m x) - \frac{f_a(2^n x)}{2^n} \right\|
\]
\[
\leq \frac{1}{2} \sum_{k=0}^{n-1} \frac{\alpha(2^{k+m} x, 2^{k+m} x)}{2^{k+m}}
\]
\[
\leq \frac{1}{2} \sum_{k=0}^{\infty} \frac{\alpha(2^{k+m} x, 2^{k+m} x)}{2^{k+m}}
\]
\[\to 0 \text{ as } m \to \infty\]
for all \( x \in X \). Hence the sequence \( \left\{ \frac{f_a(2^n x)}{2^n} \right\} \) is Cauchy sequence. Since \( Y \) is complete, there exists a mapping \( A : X \to Y \) such that
\[
A(x) = \lim_{n \to \infty} \frac{f_a(2^n x)}{2^n} \quad \forall x \in X.
\]
Letting \( n \to \infty \) in (11) we see that (6) holds for all \( x \in X \). To prove that \( A \) satisfies (1), replacing \( (x, y) \) by \( (2^m x, 2^m y) \) and dividing by \( 2^m \) in (4), we obtain
\[
\frac{1}{2^n} \left\| f_a(2^n x + 2^n y) + f_a(2^n x - 2^n y) - 2f_a(2^n x) + f_a(2^n y) + f_a(-2^n y) \right\| \leq \frac{1}{2^n} \alpha(2^n x, 2^n y)
\]
for all \( x, y \in X \). Letting \( n \to \infty \) in the above inequality and using the definition of \( A(x) \), we see that
\[
A(x + y) + A(x - y) = 2A(x) + A(y) + A(-y).
\]
Hence \( A \) satisfies (1) for all \( x, y \in X \). It follows from (5) that
\[
\| A(xy) - A(x)y - xA(y) \|
\]
\[
= \frac{1}{2^m} \| f_a(2^m (xy)) - f_a(2^m x)(2^m y) - (2^m x)f_a(2^m y) \|
\]
\[
\leq \frac{1}{2^n} \beta(2^n x, 2^n y)
\]
\[\to 0 \text{ as } n \to \infty\]
for all \(x, y \in X\). To prove that \(A\) is unique, let \(B(x)\) be another mapping satisfying (1) and (6), then

\[
\|A(x) - B(x)\| = \frac{1}{2^n} \|A(2^n x) - B(2^n x)\|
\leq \frac{1}{2^n} \{\|A(2^n x) - f_a(2^n x)\| + \|f_a(2^n x) - B(2^n x)\|\}
\leq \sum_{k=0}^{\infty} \alpha(2^{k+n} x, 2^{k+n} x) \frac{2}{2(k+n)}
\rightarrow 0 \text{ as } n \rightarrow \infty
\]

for all \(x \in X\). Hence \(A\) is unique. Thus the mapping \(A : X \rightarrow Y\) is a unique Additive Derivation mapping satisfying (6).

For \(j = -1\), we can prove a similar stability result. This completes the proof of the theorem. \(\square\)

The following Corollary is an immediate consequence of Theorem 2.1 concerning the stability of (1).

**Corollary 2.2.** Let \(f_a : X \rightarrow Y\) be a odd mapping and there exists real numbers \(\lambda\) and \(s\) such that

\[
\|f_a(x + y) + f_a(x - y) - 2f_a(x) - f_a(y) - f_a(-y)\|
\leq \begin{cases} 
\lambda, & s < 1 \text{ or } s > 1; \\
\lambda \| |x|^s + |y|^s \|, & s < \frac{1}{2} \text{ or } s > \frac{1}{2}; \\
\lambda \{ |x|^s |y|^s + \{|x|^{2s} + |y|^{2s}\} \}, & s < \frac{1}{2} \text{ or } s > \frac{1}{2}; 
\end{cases}
\tag{12}
\]

\[
\|f_a(xy) - f_a(x)y - xf_a(y)\|
\leq \begin{cases} 
\lambda, & s < 1 \text{ or } s > 1; \\
\lambda |x|^s |y|^s, & s < \frac{1}{2} \text{ or } s > \frac{1}{2}; \\
\lambda \{ |x|^s |y|^s + \{|x|^{2s} + |y|^{2s}\} \}, & s < \frac{1}{2} \text{ or } s > \frac{1}{2}; 
\end{cases}
\tag{13}
\]

for all \(x, y \in X\). Then there exists a unique Additive Derivation function \(A : X \rightarrow Y\) such that

\[
\|f_a(x) - A(x)\| \leq \begin{cases} 
\lambda, & 2\lambda |x|^s \\
\frac{|x|^{2s}}{2 - 2^s}, & 3\lambda |x|^{2s} \\
\frac{|x|^{2s}}{2 - 2^{2s}}, & |x|^{2s} \\
\end{cases}
\tag{14}
\]

for all \(x \in X\).
3 Stability Results: Quadratic Derivations

In this section, the author establish the generalized Ulam-Hyers stability of quadratic derivations of the AQ-functional equation (1).

Definition 3.1. Quadratic Derivation. A C-linear mapping Q : X → X is called Quadratic Derivation on X if Q satisfies

\[ Q(xy) = Q(x)y^2 + x^2Q(y) \]  

for all \( x, y \in X \).

Theorem 3.1. Let \( j = \pm 1 \). Let \( f_q : X \to Y \) be a even mapping for which there exist a function \( \alpha, \beta : X^2 \to [0, \infty) \) with the condition

\[ \sum_{n=0}^{\infty} \alpha \left( \frac{2^n j x, 2^n j y}{2^{2nj}} \right) \text{ converges in } \mathbb{R} \text{ and } \lim_{n \to \infty} \frac{\alpha (2^n j x, 2^n j y)}{2^{2nj}} = 0 \]  

\[ \sum_{n=0}^{\infty} \beta \left( \frac{2^n j x, 2^n j y}{2^{4nj}} \right) \text{ converges in } \mathbb{R} \text{ and } \lim_{n \to \infty} \frac{\beta (2^n j x, 2^n j y)}{2^{4nj}} = 0 \]

such that the functional inequalities

\[ \| f_q(x + y) + f_q(x - y) - 2f_q(0) - f_q(y) - f_q(-y) \| \leq \alpha(x, y) \]

and

\[ \| f_q(xy) - x^2 f_q(y) - f_q(x)y^2 \| \leq \beta(x, y) \]

for all \( x, y \in X \). Then there exists a unique Quadratic Derivation mapping \( Q : X \to Y \) satisfying the functional equation (1) and

\[ \| f_q(x) - Q(x) \| \leq \frac{1}{4} \sum_{k=0}^{\infty} \frac{\alpha(2^k j x, 2^k j x)}{2^{2kj}} \]

for all \( x \in X \). The mapping \( Q(x) \) is defined by

\[ Q(x) = \lim_{n \to \infty} \frac{f_q(2^n j x)}{2^{2nj}} \]

for all \( x \in X \).

Proof. It follows from (5) that

\[ \| Q(xy) - x^2 Q(y) - Q(x)y^2 \| \]

\[ = \frac{1}{2^{2nj}} \| f_q(2^n(xy)) - (2^n x)^2 f_q(2^n y) - f_q(2^n x)(2^n y)^2 \| \]

\[ \leq \frac{1}{2^{2nj}} \beta (2^n x, 2^n y) \]

\[ \to 0 \text{ as } n \to \infty \]
for all \( x, y \in X \). The rest of the proof is similar tracing to that of Theorem 2.1. Thus the mapping 
\[ Q : X \to Y \] 
is a unique Quadratic Derivation mapping satisfying (6).

The following Corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1).

**Corollary 3.2.** Let \( f_q : X \to Y \) be a even mapping and there exists real numbers \( \lambda \) and \( s \) such that

\[
\|f_q(x + y) + f_q(x - y) - 2f_q(x) - f_q(y) - f_q(-y)\| \leq \begin{cases} 
\lambda, & s < 2 \text{ or } s > 2; \\
\lambda \|x\|^s + \|y\|^s, & s < 1 \text{ or } s > 1; \\
\lambda \{\|x\|^s \|y\|^s + \{|x|^{2s} + |y|^{2s}\}\}, & s < 1 \text{ or } s > 1;
\end{cases}
\] (8)

for all \( x, y \in X \). Then there exists a unique quadratic Deviation function \( Q : X \to Y \) such that

\[
\|f_q(x) - x^2f_q(y) - f_q(x)y^2\| \leq \begin{cases} 
\lambda, & s < 2 \text{ or } s > 2; \\
\lambda \|x\|^s \|y\|^s, & s < 1 \text{ or } s > 1; \\
\lambda \{\|x\|^s \|y\|^s + \{|x|^{2s} + |y|^{2s}\}\}, & s < 1 \text{ or } s > 1;
\end{cases}
\] (9)

for all \( x, y \in X \). Then there exists a unique quadratic Deviation function \( Q : X \to Y \) such that

\[
\|f_q(x) - x^2f_q(y) - f_q(x)y^2\| \leq \begin{cases} 
\lambda, & s > 2; \\
\lambda \|x\|^s \|y\|^s, & s < 1; \\
\lambda \{\|x\|^s \|y\|^s + \{|x|^{2s} + |y|^{2s}\}\}, & s < 1;
\end{cases}
\] (10)

for all \( x \in X \).

4 Stability Results: Mixed Derivations

In this section, the author present the generalized Ulam-Hyers stability of mixed derivations of the AQ-functional equation (11).

**Theorem 4.1.** Let \( \| \) = \( \pm 1 \). Let \( f : X \to Y \) be a odd mapping for which there exist a function \( \alpha, \beta : X^2 \to [0, \infty) \) with the conditions (2), (3), (2) and (3) such that the functional inequalities

\[
\|f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y)\| \leq \alpha(x, y)
\] (1)
for all $x, y \in X$. Then there exists a unique Additive Derivation mapping $A : X \to Y$ and a unique Quadratic Derivation mapping $Q : X \to Y$ satisfying the functional equation (7) and

$$
\|f(x) - A(x) - Q(x)\| \leq \frac{1}{2} \left[ \frac{1}{2} \sum_{k=1}^{\infty} \left( \frac{\alpha(2^{k+1}x, 2^{k+1}x)}{2^{k+1}} + \frac{\alpha(-2^{k+1}x, 2^{k+1}x)}{2^{k+1}} \right) + \frac{1}{4} \sum_{k=1}^{\infty} \left( \frac{\alpha(2^{k+1}x, 2^{k+1}x)}{2^{k+1}} + \frac{\alpha(-2^{k+1}x, 2^{k+1}x)}{2^{k+1}} \right) \right]
$$

(2)

for all $x \in X$. The mapping $A(x)$ and $Q(x)$ are defined in (7) and (8) respectively for all $x \in X$.

**Proof.** Let $f_0(x) = \frac{f_o(x) - f_o(-x)}{2}$ for all $x \in X$. Then $f_o(0) = 0$ and $f_o(-x) = -f_o(x)$ for all $x \in X$. Hence

$$
\|f_o(x + y) + f_o(x - y) - 2f_o(x) - f_o(y) - f_o(-y)\| \leq \frac{\alpha(x, y)}{2} + \frac{\alpha(-x, y)}{2}
$$

(3)

By Theorem 3.1 we have

$$
\|f_o(x) - A(x)\| \leq \frac{1}{4} \left[ \frac{1}{2} \sum_{k=1}^{\infty} \left( \frac{\alpha(2^{k+1}x, 2^{k+1}x)}{2^{k+1}} + \frac{\alpha(-2^{k+1}x, 2^{k+1}x)}{2^{k+1}} \right) \right]
$$

(4)

for all $x \in X$. Also, let $f_e(x) = \frac{f_e(x) + f_e(-x)}{2}$ for all $x \in X$. Then $f_e(0) = 0$ and $f_e(-x) = f_e(x)$ for all $x \in X$. Hence

$$
\|f_e(x + y) + f_e(x - y) - 2f_e(x) - f_e(y) - f_e(-y)\| \leq \frac{\alpha(x, y)}{2} + \frac{\alpha(-x, y)}{2}
$$

(5)

By Theorem 3.1 we have

$$
\|f_e(x) - Q(x)\| \leq \frac{1}{8} \sum_{k=1}^{\infty} \left( \frac{\alpha(2^{k+1}x, 2^{k+1}x)}{2^{2k+1}} + \frac{\alpha(-2^{k+1}x, 2^{k+1}x)}{2^{2k+1}} \right)
$$

(6)

for all $x \in X$. Define

$$
f(x) = f_e(x) + f_o(x)
$$

(7)

for all $x \in X$. From (4), (5) and (7), we arrive

$$
\|f(x) - A(x) - Q(x)\| = \|f_o(x) + f_o(-x) - A(x) - Q(x)\|
\leq \|f_o(x) - A(x)\| + \|f_e(x) - Q(x)\|
\leq \frac{1}{4} \sum_{k=1}^{\infty} \left( \frac{\alpha(2^{k+1}x, 2^{k+1}x)}{2^{k+1}} + \frac{\alpha(-2^{k+1}x, 2^{k+1}x)}{2^{k+1}} \right) + \frac{1}{8} \sum_{k=1}^{\infty} \left( \frac{\alpha(2^{k+1}x, 2^{k+1}x)}{2^{2k+1}} + \frac{\alpha(-2^{k+1}x, 2^{k+1}x)}{2^{2k+1}} \right)
$$

for all $x \in X$. Hence the theorem is proved. □
Using Corollaries 2.2 and 3.2 we have the following Corollary concerning the stability of (1).

**Corollary 4.1.** Let \( f : X \to Y \) be a mapping and there exists real numbers \( \lambda \) and \( s \) such that
\[
\| f(x + y) + f(x - y) - 2f(x) - f(y) - f(-y) \| \leq \begin{cases} 
\lambda, & s < 1 \text{ or } s > 1; \\
\lambda (||x||^s + ||y||^s), & s < \frac{1}{2} \text{ or } s > \frac{1}{2}; \\
\lambda ||x||^s ||y||^s, & s < \frac{1}{2} \text{ or } s > \frac{1}{2}; \\
\lambda \{ ||x||^s ||y||^s + \{ ||x||^{2s} + ||y||^{2s} \} \} , & s < \frac{1}{2} \text{ or } s > \frac{1}{2}; 
\end{cases}
\]

and (13), (9) for all \( x, y \in X \). Then there exists a unique Additive Deviation function \( A : X \to Y \) and a unique quadratic Deviation function \( Q : X \to Y \) such that
\[
\| f(x) - A(x) - Q(x) \| \leq \begin{cases} 
\lambda \left( 1 + \frac{1}{3} \right), \\
2\lambda \left( \frac{1}{2 - 2s} + \frac{1}{4 - 2s} \right) ||x||^s, \\
\lambda \left( \frac{1}{2 - 2s} + \frac{1}{4 - 2s} \right) ||x||^{2s}, \\
3\lambda \left( \frac{1}{2 - 2s} + \frac{1}{4 - 2s} \right) ||x||^{2s}
\end{cases}
\]

for all \( x \in X \).

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**References**


Discrete Almost Periodic Operators

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ABSTRACT

This paper deals with discrete almost periodic linear operators in the space of bounded sequences. We study the invertibility of such operators in that space, as well as in the space of almost periodic sequences. One of main results is a discrete version of well-known First Favard Theorem, and is based on the notion of the envelope of an almost periodic operator. Another result is restricted to finite order operators. It characterizes the invertibility in terms of the operator in question only.

RESUMEN

Este trabajo trata de operadores lineales discretos casi periódicos en el espacio de las secuencias acotadas. Estudiamos la invertibilidad de dichos operadores en ese espacio, así como en el espacio de secuencias casi periódicas. Uno de los resultados principales es una versión discreta del conocido Primer Teorema de Favard, y se basa en la noción de la envolvente de un operador casi periódico. Otro resultado se restringe a los operadores de orden finito. Se caracteriza la invertibilidad solamente en términos del operador en cuestión.

Keywords and Phrases: Almost periodic sequence, discrete operator, Favard condition.

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1 Dedicated to Professor Gaston M. N’Guérékata on the occasion of his 60th Birthday
1 Introduction

The theory of almost periodic differential equations has been initiated by J. Favard in his pioneering work [5]. Today the theory is well-developed not only for ordinary differential equations, but also for abstract evolution equations and partial differential equations. Contemporary presentations of the theory can be found in many monographs and survey articles (see, e.g., [1, 3, 7, 9, 11, 13, 14] and references therein).

Difference equations constitute a natural counterpart of the theory of differential equations. We refer to [4] for a contemporary introductory presentation of the theory of difference equations. There is a number of papers that deal with almost periodic difference equations. Most of them concern either special equations (see, e.g., [17]), or first order systems [6, 12]. In particular [12] contains certain discrete versions of First and Second Favard Theorems. To the best of our knowledge, there are only few papers dedicated to general almost periodic linear difference equations, or, equivalently, almost periodic discrete linear operators (see [2, 15, 16] and references therein). In particular, in [15] certain discrete version of the First Favard Theorem is obtained (see Corollary 5.2). This version requires coercivity estimate (10) and is not similar to typical Favard type assumptions. Thus, at the moment the theory of almost periodic difference equations is not completely parallel to the theory of almost periodic differential equations.

The aim of the present paper is to fill, at list partially, the gap mentioned above. We accept the functional analytic point of view. This means that we study almost periodic operators in the space $l^\infty$ of bounded sequences with values in a finite dimensional space and their restrictions to the space $ap$ of almost periodic sequences. We obtain certain criteria for such an operator $A$ to have a bounded inverse operator. Equivalently, the invertibility means that the equation $Ax = y$ has a unique solution $x \in l^\infty$ for every right hand side $y \in l^\infty$. One of our main result, Theorem 5.1, is an exact analogue of the version of First Favard Theorem for differential equations obtained by E. Mukhamadiev in [10] (see also [14]). The second result, Theorem 6.2, is, in a sense, dual to Theorem 5.1 but it holds for operators of finite order. This is an analogue of a result by M. Krasnosel’skii, V. Burd and Y. Kolesov [7].

The paper is organized as follows. Section 2 is a quick reminder of basic facts about almost periodicity. In Section 3 we discuss bounded linear operators in the space $l^\infty$. In particular, we introduce important concepts of $c$-convergence and $c$-continuity. The main result of the section is Proposition 3.1 which shows that any $c$-continuous operator is of the form $11$ (this is a result by V. Slyusarchuk [15]). Almost periodic operators are introduced in Section 4. Sections 5 and 6 contain our main results.

In what follows, we consider elements of sequence spaces as functions on the set of integers $\mathbb{Z}$. We use the notation $\{\cdot\}$ to list the values of such a function. On the other hand, the notation $[\cdot]$ stands for the lists of elements of a set.
2 Almost Periodic Functions and Sequences

Let $E$ be a Banach space, with the norm $\| \cdot \|_E$, over real or complex numbers. We denote by $C_b(E)$ the space of bounded continuous functions on $\mathbb{R}$ with values in $E$. This is a Banach space with the norm

$$\| f \|_{C_b} = \sup_{t \in \mathbb{R}} \| f(t) \|_E.$$ 

A function $f \in C_b(E)$ is almost periodic if the family $\{ f(\cdot + \tau) \}_{\tau \in \mathbb{R}}$ of shifts is a precompact set in $C_b(E)$. Almost periodic functions form a closed subspace $\text{AP}(E)$ of $C_b(E)$, hence, a Banach space.

By $l^\infty(E)$ we denote the space of all bounded two-sided sequences $x = [x(n)]_{n \in \mathbb{Z}}$ with values in $E$. This is a Banach space endowed with the norm

$$\| x \|_{l^\infty} = \sup_{n \in \mathbb{Z}} \| x(n) \|_E.$$ 

In what follows we also need the space of $E$-valued sequences $l^1(E)$, with the norm

$$\| x \|_{l^1} = \sum_{n \in \mathbb{Z}} \| x(n) \|_E.$$ 

In this paper we do not use other spaces $l^p$.

A sequence $x = [x(n)]_{n \in \mathbb{Z}} \in l^\infty(E)$ is almost periodic if the set of its shifts $\{ [x(\cdot + q)] \}_{q \in \mathbb{Z}}$ is a precompact set in $l^\infty(E)$. The set of all almost periodic sequences is a closed subspace $\text{ap}(E) \subset l^\infty(E)$. Hence, $\text{ap}(E)$ is a Banach space.

It is convenient to introduce operators of translation $T_q$, $q \in \mathbb{Z}$, acting in the space $l^\infty(E)$ by the formula

$$(T_q x)(n) = x(n + q), \quad n \in \mathbb{Z}.$$ 

These are linear bounded operators. Moreover, they are isometric operators, i.e.,

$$\| T_q x \|_{l^\infty} = \| x \|_{l^\infty}, \quad q \in \mathbb{Z}.$$ 

The operators $T_q$ form a one-parameter discrete group of operators, i.e.,

(i) $T_{q_1 + q_2} = T_{q_1} T_{q_2}, \quad q_1, q_2 \in \mathbb{Z},$

(ii) $T_0 = I,$

(iii) $T_{-q} = T_q^{-1}, \quad q \in \mathbb{Z},$

where $I$ stands for the identity operator. In terms of these operators the almost periodicity of a sequence $x \in l^\infty$ means that the set $\{ T_q x \}_{q \in \mathbb{Z}}$ is precompact in $l^\infty(E)$.

The following simple statement is well-known (see, e.g., [3, Theorem 1.27]) and clarifies a relation between almost periodic sequences and functions.
**Proposition 2.1.** The restriction operator \( R : C_b(E) \to l^\infty(E) \) defined by \( f \mapsto [f(n)]_{n \in \mathbb{Z}} \) maps \( \text{AP}(E) \) onto \( \text{ap}(E) \). Furthermore, there exists a linear operator \( \tilde{J} : E \to C_b(E) \), an extension operator, such that \( (\tilde{J}x)(n) = x(n) \) for all \( n \in \mathbb{Z} \), \( \|\tilde{J}x\|_{C_b} = \|x\|_{l^\infty} \) for all \( x \in l^\infty(E) \), and \( \tilde{J}(\text{ap}(E)) \subseteq \text{AP}(E) \).

Making use of Proposition 2.1 one can transfer many results about almost periodic functions to almost periodic sequences.

Finally, we introduce a simple, but important, notion of periodization. Given a positive integer \( j \), let
\[
Q_j = \{ n \in \mathbb{Z} | j - 1 \leq n \leq j \}.
\]
For any \( x \in l^\infty(E) \), its \( 2j \)-periodization is a \( 2j \)-periodic sequence, say \( x_j = [x_j(n)]_{n \in \mathbb{Z}} \), such that \( x_j(n) = x(n) \) for all \( n \in Q_j \). Obviously, \( x_j \xrightarrow{c} x \) and, hence, the space \( \text{ap}(E) \) is \( c \)-dense in \( l^\infty(E) \).

## 3 Linear Operators in \( l^\infty \)

In the rest of the paper \( E \) stands for a finite dimensional Banach space. For sequences in the space \( l^\infty(E) \) one can introduce several kinds of convergence. In this paper we use the standard convergence with respect to the norm of \( l^\infty(E) \) and the so-called \( c \)-convergence. A sequence \( x_k \in l^\infty(E) \) \( c \)-converges to \( x \in l^\infty(E) \) (in symbols \( x_k \xrightarrow{c} x \)) if the sequence \( x_k \) is bounded in \( l^\infty(E) \) and \( x_k(n) \to x(n) \) for all \( n \in \mathbb{Z} \). In this case we also write \( x = \text{c-lim} x_k \).

By \( L(l^\infty(E)) \) we denote the Banach algebra of all bounded linear operators in \( l^\infty(E) \). An operator \( A \in L(l^\infty(E)) \) is \( c \)-continuous if for any sequence \( x_k \in l^\infty(E) \) such that \( x_k \xrightarrow{c} x \) we have that \( Ax_k \xrightarrow{c} Ax \). The set of all \( c \)-continuous operators is a closed subalgebra of the Banach algebra \( L(l^\infty(E)) \) (see [2] Proposition 1). We denote this subalgebra by \( L_c(l^\infty(E)) \).

We consider operators of the form
\[
(Ax)(n) = \sum_{m \in \mathbb{Z}} A(n, m)x(m), \quad n \in \mathbb{Z},
\]
where \( A(n, m) \in L(E) \). The double sequence \( A = [A(n, m)]_{n, m \in \mathbb{Z}} \) is called the kernel of \( A \). Given such a kernel \( A \), we set
\[
\|A\| = \sup_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \|A(n, m)\|_{L(E)}.
\]
An alternative representation of operator (1) is
\[
(Ax)(n) = \sum_{k \in \mathbb{Z}} a(n, k)x(n + k), \quad n \in \mathbb{Z},
\]
where \( a(n, k) = A(n, n + k) \) are called the coefficients of \( A \). It is easily seen that, for double sequences \( A \) and \( a = [a(n, m)]_{n, m \in \mathbb{Z}} \), we have \( \|A\| = \|a\| \). Such sequences can be considered as
sequences indexed by \( n \) with values in the space of sequences indexed by \( m \). From this point of view the norm defined by (2) is exactly the \( l^\infty(\ell^1(\mathbb{E})) \)-norm.

The nontrivial part of the next result goes back to [16]. Since this paper is not available in English, we present the proof here.

**Proposition 3.1.** A linear operator \( A \) in \( l^\infty(\mathbb{E}) \) is a bounded \( c \)-continuous operator if and only if \( A \) is of the form (1) with \( \|A\| < \infty \). Moreover, in this case

\[
\|A\|_{L(l^\infty(\mathbb{E}))} = \|A\|.
\]

**Proof.** Suppose that \( \|A\| < \infty \). Then it is well-known, and easily seen, that

\[
\|A\|_{L(l^\infty(\mathbb{E}))} \leq \|A\|.
\]

Hence, \( A \in L(l^\infty(\mathbb{E})) \).

To prove that \( A \) is \( c \)-continuous, suppose that \( x_k \xrightarrow{c} 0 \). Obviously, the sequence \( Ax_k \) is bounded in \( l^\infty(\mathbb{E}) \), and we need to show that \( (Ax_k)(n) \to 0 \) in \( \mathbb{E} \) for every \( n \in \mathbb{Z} \). For any positive \( N \in \mathbb{Z} \)

\[
\|(Ax_k)(n)\|_\mathbb{E} \leq \left( \sum_{|m| \leq N} + \sum_{|m| > N} \right) \|A(n, m)\|_{\ell^1(\mathbb{E})}\|x_k(m)\|_\mathbb{E}.
\]

Since the sequence \( x_k \) is bounded in \( l^\infty(\mathbb{E}) \) and \( \|A\| < \infty \), choosing \( N \) large enough we can make the second sum in the right-hand side sufficiently small. Next, since \( x_k \xrightarrow{c} 0 \), the first sum is sufficiently small provided \( k \) is large enough. This proves the first statement.

Suppose that \( A \in L_c(l^\infty(\mathbb{E})) \). First, we define its kernel as follows. Denote by \( J_m : \mathbb{E} \to \ell^\infty(\mathbb{E}) \), \( m \in \mathbb{Z} \), the operator defined by

\[
(J_m u)(n) = \begin{cases} u & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}.
\]

The operator \( P_n : l^\infty(\mathbb{E}) \to \mathbb{E} \), \( n \in \mathbb{Z} \), is defined by \( P_n x = x(n) \). For all \( n, m \in \mathbb{Z} \), we set

\[
A(n, m) = P_n A J_m.
\]

Obviously, \( A(n, m) \in L(\mathbb{E}) \).

Now we prove (1), where the right-hand side converges in \( l^\infty(\mathbb{E}) \) for each \( n \in \mathbb{Z} \). Indeed, given \( x \in l^\infty(\mathbb{E}) \), we set

\[
x_k(n) = \begin{cases} x(n) & \text{if } |n| \leq k \\ 0 & \text{otherwise} \end{cases}.
\]

It is easily seen that \( x_k \xrightarrow{c} x \). Since \( A \) is \( c \)-continuous, then for every \( n \in \mathbb{Z} \)

\[
(Ax_k)(n) = \sum_{|m| \leq k} A(n, m)x(m) \to (Ax)(n)
\]
as required.

Let us prove that
\[ \|A\| \leq \|A\|_{L(1^\infty(E))}. \]
For each \( k \in \mathbb{Z} \), we consider an element \( x_k \in 1^\infty(E) \) such that \( \|x_k(n)\|_E = 1 \) and
\[ \|A(k,n)x_k(n)\|_E = \|A(k,n)\|_{L(E)} \]
for all \( n \in \mathbb{Z} \). Since \( \dim E < \infty \), such a sequence exists. Then
\[ \sum_{m \in \mathbb{Z}} \|A(k,m)\|_{L(E)} = \|(A x_k)(k)\|_E \leq \|A x_k\|_{1^\infty} \leq \|A\|_{L(1^\infty)} \]
because \( \|x_k\|_{1^\infty(E)} = 1 \). This implies the required.

\[ \square \]

Proposition 3.1 and [2, Proposition 3] imply immediately

**Corollary 3.2.** Suppose that \( A \in L_c(1^\infty(E)) \) has a bounded inverse operator. Then the inverse operator is \( c \)-continuous and, hence, is of the form
\[ (A^{-1}x)(n) = \sum_{m \in \mathbb{Z}} G(n,m)x(m), \quad n \in \mathbb{Z}, \]
with \( \|G\| = \|A^{-1}\|_{L(1^\infty(E))} < \infty. \)

The kernel \( G \) in Corollary 3.2 is often called the Green function of the operator \( A^{-1} \).

**Remark 3.3.** Under the assumption of Corollary 3.2, suppose in addition that the kernel \( A \) satisfies
\[ \|A(m,n)\|_{L(E)} \leq \frac{c}{(1+|n-m|)^\alpha}, \quad m, n \in \mathbb{Z}, \quad (4) \]
with \( c > 0 \) and \( \alpha > 2 \). Then for every \( \theta > 0 \) small enough there exists \( c_\theta > 0 \) such that the Green function satisfies
\[ \|G(m,n)\|_{L(E)} \leq \frac{c_\theta}{(1+|n-m|)^{\alpha-1-\theta}}, \quad m, n \in \mathbb{Z}. \quad (5) \]
Furthermore, if
\[ \|A(m,n)\|_{L(E)} \leq c \exp(-\delta |n-m|), \quad m, n \in \mathbb{Z}, \quad (6) \]
with \( c > 0 \) and \( \delta > 0 \), then there exist \( c_1 > 0 \) and \( \varepsilon > 0 \) such that
\[ \|G(m,n)\|_{L(E)} \leq c_1 \exp(-\varepsilon |n-m|), \quad m, n \in \mathbb{Z}, \quad (7) \]
(see [2] and [17]).
4 Almost Periodic Operators

We say that $A \in L(l^\infty(E))$ is an almost periodic operator if the sequence of operators $[T_qAT_{-q}]_{q \in \mathbb{Z}}$ is an almost periodic sequence with values in $L(l^\infty(E))$.

**Proposition 4.1.** An operator $A \in L(l^\infty(E))$ is almost periodic if and only if the set $\{T_qAT_{-q}\}_{q \in \mathbb{Z}}$ is precompact in $L(l^\infty(E))$.

For the proof we refer to [2, Proposition 6].

The envelope, or hull, $H(A)$ of an almost periodic operator $A \in L(l^\infty(E))$ is the closure of the set $\{T_qAT_{-q}\}_{q \in \mathbb{Z}}$ in the space $L(l^\infty(E))$. This is a compact set.

Now we collect some properties of almost periodic operators obtained in [2].

**Proposition 4.2.** Suppose that $A \in L(l^\infty(E))$ is almost periodic operator. Then the following statements hold:

(a) $A(ap(E)) \subset ap(E)$.

(b) If $A$ has a bounded inverse operator, then $A^{-1}$ is an almost periodic operator and, hence, $A_{ap(E)}$ has a bounded inverse operator in $L(ap(E))$. Moreover, all operators in the envelope $H(A)$ are invertible and

$$H(A^{-1}) = \{\tilde{A}^{-1} : \tilde{A} \in H(A)\}.$$ 

(c) If, in addition, $A$ is of the form (1), the kernel $A$ satisfies (4) with $c > 0$ and $\alpha > 2$, and if $A_{ap(E)}$ has a bounded inverse operator in $L(ap(E))$, then the operator $A$ has a bounded inverse operator in $L(l^\infty(E))$.

**Remark 4.3.** Actually, under the assumptions of Proposition 4.2 (c) the operator $A$ is $c$-continuous.

**Proposition 4.4.** Suppose that $A \in L(l^\infty(E))$ is a $c$-continuous operator of the form (1). The following statements are equivalent:

(i) The operator $A$ is almost periodic.

(ii) The kernel $A$ is almost periodic along the diagonal with respect to norm (5), i.e., the sequence of kernels $[A(\cdot + q, \cdot + q)]_{q \in \mathbb{Z}}$ is an almost periodic sequence in with values in the space of kernels endowed with the norm $\| \cdot \|$.

(iii) The sequence $[a(n, \cdot)]_{n \in \mathbb{Z}}$ of coefficients is an almost periodic sequence with values in $L^1(L(E))$.

**Proof.** A straightforward verification shows that $[A(n + q, m + q)]$ is the kernel of the operator $T_qAT_{-q}$. Now the required follows immediately from Proposition 3.1 and remarks after equation (3).
Remark 4.5. If an almost periodic operator $A \in L^1(\mathbb{E})$ is of the form $\{A\}$, then every operator $\tilde{A} \in H(A)$ is of the same form. The kernel of $\tilde{A}$ is a limit point of the set $\{A(\cdot + q, \cdot + q)\}_{q \in \mathbb{Z}}$ in the space of kernels with respect to the norm $\|\cdot\|$, while the collection of the coefficients is the limit point of the set $\{a(\cdot + q, \cdot)\}_{q \in \mathbb{Z}}$ in the space $L^1(\mathbb{E})$.

5 Favard Type Theorem

In this section we prove the following result of Favard type for almost periodic operators in $L^\infty(\mathbb{E})$.

Theorem 5.1. An almost periodic operator $A \in L_\alpha(\mathbb{E})$ has a bounded inverse operator if and only if the following condition is satisfied:

$$\text{(F) every operator in the envelope } H(A) \text{ is injective.}$$

Proof. If $A \in L_\alpha(\mathbb{E})$ is invertible, then, by Proposition 4.2, all operators in the envelope are invertible, and (F) follows.

Now suppose that (F) is satisfied. To prove that the operator $A$ has a bounded inverse, it is enough to show that for any $y \in L^\infty(\mathbb{E})$ there exists $x \in L^\infty(\mathbb{E})$ such that

$$Ax = y. \quad (8)$$

For any positive integer $j$, denote by $x_j = [x_j(n)]_{n \in \mathbb{Z}}$ the 2j-periodization of $x$. Similarly, we denote by $a_j = [a_j(n, k)]_{n, k \in \mathbb{Z}}$ the 2j-periodization of $a = [a(n, k)]_{n, k \in \mathbb{Z}}$ with respect to the variable $n$. According to equation (3), $a_j$ generates an operator $A_j$ that belongs to $L_\alpha(\mathbb{E})$. Notice that 2j-periodic sequences form a finite dimensional subspace of $L^\infty(\mathbb{E})$, and $A_j$ leaves that subspace invariant.

We solve the equation

$$A_jx_j = y_j \quad (9)$$

in the subspace of 2j-periodic sequences provided $j$ is large enough. Since the problem is finite dimensional, it is sufficient to show that the associated homogeneous problem has only zero solution. Assuming the contrary, one can find a sequence $j_l \to \infty$ such that $A_{j_l}x_{j_l} = 0$ for some $x_{j_l} \in L^\infty(\mathbb{E})$ with $\|x_{j_l}\|_{L^\infty} = 1$. Then there exists $q_1 \in Q_{j_l}$ such that $\|x_{j_l}(q_1)\|_{\mathbb{E}} = 1$. We put $\tilde{x}_1 = T_{q_1}x_{j_l}$ and $A_{j_l} = T_{q_1}A_{j_l}T_{-q_1}$. Then $\|\tilde{x}_1\|_{L^\infty} = 1$, $\|\tilde{x}_1(0)\|_{\mathbb{E}} = 1$, and $A_{j_l}\tilde{x}_1 = 0$. Passing to a subsequence, we can assume that $\tilde{x}_1 \to \tilde{x}$ and

$$A_l = T_{q_1}AT_{-q_1} \to \tilde{A}$$

in $L(\mathbb{E})$. Obviously, $\|\tilde{x}(0)\|_{\mathbb{E}} = 1$ and $\tilde{A} \in H(A)$. Given $n \in \mathbb{Z}$ we have

$$(\tilde{A}\tilde{x})(n) = ([\tilde{A}\tilde{x}](n) - (\tilde{A}\tilde{x}_1)(n)) + ([\tilde{A}\tilde{x}_1](n) - (A_l\tilde{x}_1)(n)) + (A_l\tilde{x}_1)(n).$$
Here the first term in the right-hand side tends to 0 since the operator $\tilde{A}$ is $c$-continuous,

$$|| (\tilde{A} \tilde{x}_l)(n) - (A_l \tilde{x}_l)(n)||_E \leq ||\tilde{A} - A_l||_{L(l^\infty(E))} \to 0,$$

and

$$(A_l \tilde{x}_l)(n) = [(T_{q_1}AT_{-q_1})T_{q_1}x_l](n) = (T_{q_1}Ax_l)(n) = (T_{q_1}A_lx_l)(n) = 0$$

provided $|q_1| \geq |n|$. Hence, $\tilde{A}x = 0$, which contradicts to condition (F). Thus, equation (9) has a unique $2j$-periodic solution $x_j$ for all sufficiently large $j$.

The sequence $x_j$ is bounded in the space $l^\infty(E)$. For if this is not so, we can find a subsequence $x_{j_l}$ such that $||x_{j_l}||_{l^\infty} \to \infty$. Set

$$z_l = x_{j_l}/||x_{j_l}||_{l^\infty}.$$

Then

$$A_lz_l = y_{j_l}/||x_{j_l}||_{l^\infty}.$$  

Arguing exactly as above, we obtain that there is a nonzero $x \in l^\infty(E)$ such that $\tilde{A}x = 0$, and we arrive at contradiction to condition (F).

Now since the sequence $x_j$ is bounded in $l^\infty(E)$, then $x_j \to x$ along a subsequence. It is easy to see that $Ax = y$.

The following result is obtained in [15].

**Corollary 5.2.** Let $A \in L_c(l^\infty(E))$ be an almost periodic operator. If there exists a constant $c_0 > 0$ such that

$$||Ax||_{l^\infty} \geq c_0 ||x||_{l^\infty}$$  

for all $x \in l^\infty(E)$, then the operator $A$ is invertible in $l^\infty(E)$.

**Proof.** Since the operators $T_q$ are isometric, then, by the definition of the envelope, all operators in $E(A)$ satisfy inequality (10). This implies condition (F), and we conclude.

By Proposition [12] (b), we obtain the following

**Corollary 5.3.** If an almost periodic operator $A \in L_c(l^\infty(E))$ satisfies condition (F), then the operator $A|_{ap(E)}$ has a bounded inverse operator in the space $ap(E)$.

Furthermore, Proposition [12] (c) implies

**Corollary 5.4.** Suppose that the kernel $A$ of an almost periodic operator $A \in L_c(l^\infty(E))$ satisfies inequality (4) with $c > 0$ and $\alpha > 2$. Then the following statements are equivalent:

(i) The operator $A$ satisfies condition (F);
The operator \( A \) has a bounded inverse operator in the space \( l^\infty(E) \);

The operator \( A_{\text{ap}(E)} \) has a bounded inverse operator in the space \( \text{ap}(E) \).

6 Operators of Finite Order

Now, under certain additional assumptions, we obtain yet another criterion for an almost periodic operator to be invertible. This result is, in a sense, dual to Theorem 5.1.

In this section we consider operators of finite order. These are of the form

\[
(Ax)(n) = \sum_{k=k_1}^{k_2} a(n, k)x(n + k), \quad n \in \mathbb{Z},
\]

where \( a(n, k_1) \) and \( a(n, k_2) \) are non-zero operators in \( E \). The number \( k_2 - k_1 \) is called the order of \( A \). In what follows we always suppose that the order of \( A \) is greater than zero. The kernel \([A(n, m)]\) of \( A \) vanishes outside the strip \( \{(n, m) : k_1 \leq m - n \leq k_2\} \).

We impose the following assumptions:

(A1) \( \sup\{\|a(n, k)\|_{L(E)} : n \in \mathbb{Z}, \ k_1 \leq k \leq k_2\} < \infty \);

(A2) For all \( n \in \mathbb{Z} \) the operators \( a(n, k_1) \) and \( a(n, k_2) \) are invertible in \( L(E) \), and there exists a constant \( C > 0 \) independent of \( n \) such that

\[
\|a^{-1}(n, k_1)\|_{L(E)} \leq C
\]

and

\[
\|a^{-1}(n, k_2)\|_{L(E)} \leq C.
\]

Assumption (A1) is necessary and sufficient for an operator \( A \) of the form (11) to be a bounded linear operator in \( l^\infty(E) \). In this case, \( A \) is \( c \)-continuous automatically. Assumption (A2) is natural because it is necessary for the existence of bounded inverse operator \( A^{-1} \). Let us also mention that the operator \( A \) is almost periodic if and only if for any \( k, k_1 \leq k \leq k_2 \), the sequence \( [a(n, k)]_{n \in \mathbb{Z}} \) is almost periodic. Furthermore, the envelope \( H(A) \) of any almost periodic operator of finite order consists of operators of finite order.

The following simple, but important, property is well-known.

**Proposition 6.1.** Assume that an operator \( A \) of finite order \( \geq 1 \) satisfies (A1) and (A2). Then its null space

\[
\{x \in l^\infty(E) \mid Ax = 0\}
\]

is finite dimensional.
Proof. Let $d \geq 1$ be the order of $A$. Assumption (A2) implies immediately that the linear mapping from the null space into the space $E^d$ defined by
\[ x = [x(n)]_{n \in \mathbb{Z}} \mapsto (x(k_1), x(k_1 + 1), \ldots, x(k_2 - 1)) \]
is one-to-one.

The main result of the section is the following.

**Theorem 6.2.** Suppose that an operator $A$ of the form (11) is almost periodic, of order $d \geq 1$, and satisfies assumptions (A1) and (A2). Then the following statements are equivalent:

(i) The range of operator $A$ contains $ap(E)$;

(ii) The operator $A$ has an inverse operator in $L(l^\infty(E))$;

(iii) The operator $A|_{ap(E)}$ has an inverse operator in $L(ap(E))$.

Proof. The equivalence of (ii) and (iii) follows from Proposition 4.2. Obviously, (ii) implies (i).

Now we prove that (i) implies (ii). Assuming (i), we have to show that the equation
\[ Ax = y \quad (12) \]
has a unique solution $x \in l^\infty(E)$ for any $y \in l^\infty(E)$.

**Claim 1.** For any $y \in ap(E)$ there exists a solution $x \in ap(E)$ of equation (12) such that
\[ \|x\|_{l^\infty} \leq C\|y\|_{l^\infty}, \quad (13) \]
with some constant $C > 0$ independent of $y$.

Denote by $V$ the preimage of $ap(U)$ under the operator $A$. Since $ap(E)$ is a closed subspace of $l^\infty(E)$ and $A$ is a bounded operator, $V$ is a closed subspace as well, hence, a Banach space. The operator $A|_V$ maps $V$ onto $ap(E)$. Now the required is a particular case of a well-known result about linear operators from a Banach space onto a Banach space (for an excellent presentation see [8]).

**Claim 2.** For any $y \in l^\infty(E)$ there exists a solution $x \in l^\infty(E)$ of equation (12) that satisfies estimate (13).

Let $y_j$ be the $2j$-periodization of $y$. Obviously, $\|y_j\|_{l^\infty} \leq \|y\|_{l^\infty}$. By Claim 1, there exists an almost periodic solution $x_j$ of equation (12), with $y$ replaced by $y_j$, and
\[ \|x_j\|_{l^\infty} \leq C\|y_j\|_{l^\infty} \leq C\|y\|_{l^\infty}. \]
Hence, along a subsequence, \( x_j \xrightarrow{c} x \), and \( x \) is a solution of (12) that satisfies (13).

**Claim 3.** Each operator \( \tilde{\mathcal{A}} \in H(\mathcal{A}) \) maps \( l^\infty(E) \) onto \( l^\infty(E) \).

Let

\[
\tilde{\mathcal{A}} = \lim T_{q_j} \mathcal{A} T_{-q_j} \in H(\mathcal{A}).
\]

By Claim 2, given \( y \in l^\infty(E) \) there exists \( x_j \in l^\infty(E) \) such that \( \mathcal{A} x_j = T_{-q_j} y \) and

\[
\| x_j \|_{l^\infty} \leq C \| T_{-q_j} y \|_{l^\infty} = C \| y \|_{l^\infty}.
\]

Setting \( \tilde{x}_j = T_{q_j} x_j \), we have that

\[
T_{q_j} \mathcal{A} T_{-q_j} \tilde{x}_j = y. \tag{14}
\]

Passing to a subsequence, we can suppose that there exists \( x \in l^\infty \) such that \( \tilde{x}_j \xrightarrow{c} x \). Passing to the limit in equation (14), we see that \( \tilde{\mathcal{A}} x = y \).

**Claim 4.** If \( y \in \text{ap}(E) \), then every solution \( x \in l^\infty(E) \) of equation (12) is almost periodic.

Due to Claim 1, it is enough to show that every solution of the homogeneous equation, i.e., with \( y = 0 \), is almost periodic. Assume the contrary. Then there exists \( x \in l^\infty \) such that \( \mathcal{A} x = 0 \) and \( x \) is not almost periodic, i.e., the family of shifts \( \{ T_q x \}_{q \in \mathbb{Z}} \) is not precompact. Then there exist \( \varepsilon_0 > 0 \) and an infinite set of integers \( \{ q_j \} \) such that

\[
\| x_j - x_i \|_{l^\infty} \geq \varepsilon_0
\]

for \( i \neq j \), where \( x_j = T_{q_j} x \). Without loss of generality, we can suppose that

\[
T_{q_j} \mathcal{A} T_{-q_j} \rightarrow \mathcal{A} \in H(\mathcal{A})
\]

in the space \( L(l^\infty(E)) \). Since

\[
\| \tilde{\mathcal{A}} x_j \|_{l^\infty} = \| (\tilde{\mathcal{A}} - T_{q_j} \mathcal{A} T_{-q_j}) x_j \|_{l^\infty} \leq \| \mathcal{A} T_{q_j} \mathcal{A} T_{-q_j} \|_{L(l^\infty(E))} \| x \|_{l^\infty},
\]

then

\[
\tilde{\mathcal{A}} x_j \rightarrow 0 \tag{16}
\]

in \( l^\infty(E) \) as \( j \rightarrow \infty \).

Let \( \tilde{V}_0 \) be the null space of the operator \( \tilde{\mathcal{A}} \). By Proposition 6.1 this is a finite dimensional subspace in \( l^\infty(E) \). Hence, there exists a bounded projector \( \tilde{P}_0 \) in \( l^\infty(E) \) onto \( \tilde{V}_0 \). Set \( \tilde{P}_1 = I - \tilde{P}_0 \) and \( \tilde{V}_1 = \tilde{P}_1(l^\infty(E)) \). Obviously, \( \tilde{A} \tilde{P}_0 x_j = 0 \). Hence, by (16),

\[
\tilde{A} \tilde{P}_1 x_j \rightarrow 0 \tag{17}
\]

in \( l^\infty(E) \) as \( j \rightarrow \infty \).
The restriction $\tilde{A}_{V_1}$ is one-to-one and, by Claim 3, maps $V_1$ onto $l^\infty(E)$. Hence, by (17),

$$\tilde{P}_1 x_j \to 0$$

(18)
in $l^\infty(E)$ as $j \to \infty$. By the triangle inequality,

$$\|\tilde{P}_0 x_j - \tilde{P}_0 x_i\|_{l^\infty} \geq \| x_j - x_i \|_{l^\infty} - \| \tilde{P}_1 x_j \|_{l^\infty} - \| \tilde{P}_1 x_i \|_{l^\infty}.$$ 

Now (15) and (18) imply that for any $\varepsilon_1 \in (0, \varepsilon_0)$

$$\|\tilde{P}_0 x_j - \tilde{P}_0 x_i\|_{l^\infty} \geq \varepsilon_1,$$

whenever both $j$ and $i$ are large enough. Hence, the set $\{\tilde{P}_0 x_j\}$ is not a precompact set.

On the other side, $\{\tilde{P}_0 x_j\}$ is a bounded subset of a finite dimensional space $\tilde{V}_0$. Hence, it is precompact, and we arrive at a contradiction.

Claim 5. The null space $V_0$ of the operator $A$ is trivial.

Without loss of generality, we can assume that $k_2 = 1$. The restriction operator $R : l^\infty(E) \to E^d$ is defined by

$$R : x = [x(n)]_{n \in \mathbb{Z}} \mapsto (x(k_1), x(k_1 + 1), \ldots, x(0)).$$

As we have mentioned in the proof of Proposition 6.1, $R$ maps $V_0$ into $E^d$ in one-to-one manner. We set $V_0 = R(V_0)$. This is a linear subspace of $E^d$. Choose any direct complement $V_1$ to $V_0$ in $E^d$ and set

$$V_1 = \{x \in l^\infty(E) : Rx \in V_1\}.$$ 

Then $V_0 \oplus V_1 = l^\infty(E)$ and $A(V_1) = l^\infty(E)$. The operator

$$A_{|V_1} : V_1 \to l^\infty(E)$$

is one-to-one and onto and, hence, has a bounded inverse denoted by

$$B : l^\infty(E) \to V_1.$$

Now we prove that $V_0 = \{0\}$. Suppose that $Ax = 0$, and $x \neq 0$. By Claim 4, $x$ is an almost periodic sequence. Let $\theta_j = [\theta_j(n)]_{n \in \mathbb{Z}}$ be a (vector valued) sequence defined by

$$\theta_j(n) = \begin{cases} 
0 & \text{if } n \leq 0 \\
n & \text{if } 1 \leq n \leq j \\
j & \text{if } n > j.
\end{cases}$$

Consider the sequence

$$x_j = \theta_j \cdot x = [\theta_j(n)x(n)]_{n \in \mathbb{Z}}.$$
Obviously, \( x_j \in L^\infty(E) \). Moreover, \( x_j \in V_1 \) because \( x_j(n) = 0 \) if \( n \leq 0 \).

We have

\[
Ax_j = \theta_j \cdot A x + z_j = z_j,
\]

where \( z_j = [z_j(n)]_{n \in \mathbb{Z}} \) with

\[
z_j(n) = \sum_{k=k_1}^{1} (\theta_j(n+k) - \theta_j(n))a(n,k)x(n+k), \quad n \in \mathbb{Z}.
\]

It is easily seen that

\[
|\theta_j(n+k) - \theta_j(n)| \leq d
\]

for all \( n \in \mathbb{Z} \) and integer \( k \in [k_1, 1] \). Hence, \( \|z_j\|_{L^\infty} \) is bounded above by a constant independent of \( j \). Since \( x_j \in V_1 \), we have that \( x_j = Bz_j \) and, hence, \( \|x_j\|_{L^\infty} \leq C \), where \( C > 0 \) is independent on \( j \). In particular,

\[
j\|x(j)\|_{E} \leq \|x_j\|_{L^\infty} \leq C.
\]

Hence, \( x(j) \to 0 \) as \( j \to \infty \), which implies that \( x = 0 \) because, by Claim 3, \( x \) is almost periodic.

This completes the proof of the theorem.

\[\square\]

Combining Theorem 6.2 and Corollary 5.3, we obtain

**Corollary 6.3.** Under the assumptions of Theorem 6.2 the following statements are equivalent:

(i) The operator \( A \) satisfies condition (F) of Theorem 5.1.

(ii) The operator \( A \) has a bounded inverse in \( L(L^\infty(E)) \);

(iii) The operator \( A \) maps \( L^\infty(E) \) onto \( L^\infty(E) \);

(iv) The restriction \( A|_{ap(E)} \) has a bounded inverse in \( L(ap(E)) \);

(v) The restriction \( A|_{ap(E)} \) maps \( ap(E) \) onto \( ap(E) \).

**References**


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